

出版前言

2011年12月11日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长,学习他的科学思想和卓越风范,展示其丰功伟绩和人格魅力,西安交通大学举办了“纪念钱学森诞辰100周年”系列活动:作为制片方之一,参与西部电影集团摄制传记故事片《钱学森》;与中央电视台合作,出品纪录片《实验班的故事——沿着钱学森走过的路》;扩建钱学森生平业绩展馆,向校内外开放;举办钱学森科学与教育思想研讨会;出版发行《钱学森力学手稿》、《钱学森年谱(初编)》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿,这些手稿真实展示了钱学森先生博大精深的学识、开拓求实的精神和严谨奋进的作风,是钱老勇攀科学高峰和严谨治学的集中体现。这里,我们将部分原稿整理汇集成册,出版《钱学森力学手稿》,作为钱老百年诞辰的献礼。

《钱学森力学手稿》共10卷,包含两部分内容。第一部分是草稿,包括扁壳、球壳和圆柱壳屈曲分析的公式推导和数值演算。在研究圆柱壳轴压屈曲问题时,为了求得圆柱壳体的临界压力,在有关的五百多页草稿中,对多达二十多种可能的屈曲模

态逐一进行公式推演和数值计算,最终才找到满意的并在论文中采用的屈曲模态。仔细观察草稿中的数据列表,每个数字有效位数都长达八位,在手摇机械式计算机作为主要计算工具的年代,这串串数字凝聚着多少现今难以想象的艰辛劳动。

第二部分是手稿,以航空航天工程为核心,涵盖空气动力学、固体力学、火箭技术、工程控制论和物理力学等领域的部分学术论文手稿、打印稿和讲义。

《钱学森力学手稿》是在西安交通大学校领导的大力支持下,由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助,在此深表感谢。

Preliminary Calculation of
Circular Cylinder (Ⅲ)

New Calculation

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for the circular region,

$$\frac{w_0}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left\{ 1 - \left(\frac{a}{a} \right)^2 \sin^2 \theta \right\}$$

$$\frac{w}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left\{ 1 - \left(\frac{a}{a} \right)^2 \sin^2 \theta - \frac{1}{4} \left[1 - \left(\frac{a}{a} \right)^2 \right]^2 - \frac{1}{2} \left[1 - \left(\frac{a}{a} \right)^2 \right]^3 \right\}$$

$$\frac{1}{R} \frac{\partial w}{\partial R} = \frac{1}{R} \left\{ -\sin^2 \theta + 2 \left[1 - \left(\frac{a}{a} \right)^2 \right] + 6 \left[1 - \left(\frac{a}{a} \right)^2 \right]^2 \right\}$$

$$\frac{1}{R} \frac{\partial w_0}{\partial R} = \frac{1}{R} \left\{ -\sin^2 \theta \right\}$$

$$\frac{\partial^2 w}{\partial R^2} = \frac{1}{R} \left\{ -\sin^2 \theta + 2 \left[1 - 3 \left(\frac{a}{a} \right)^2 \right] + 6 \left[1 - \left(\frac{a}{a} \right)^2 \right]^2 \left[1 - 11 \left(\frac{a}{a} \right)^2 \right] \right\}$$

$$\frac{\partial^2 w_0}{\partial R^2} = \frac{1}{R} \left\{ -\sin^2 \theta \right\}$$

$$\frac{1}{R^2} \frac{\partial^2 w}{\partial t^2} = \frac{1}{R^2} \frac{\partial^2 w_0}{\partial t^2} = \frac{1}{R} \left\{ -\cos^2 \theta \right\}$$

$$\begin{aligned}
& - \left\{ \frac{1}{2} \frac{\partial^2}{\partial u^2} \frac{\partial^2}{\partial v^2} - \frac{1}{2} \frac{\partial^2}{\partial u^2} \frac{\partial^2}{\partial v^2} \right\} \\
& = \frac{1}{R^2} \left\{ (1 - \cos \theta) \left\{ 2f_1 \left[1 - 2\left(\frac{b}{a}\right)^2 \right] + 6f_2 \left[1 - \left(\frac{b}{a}\right)^2 \right] \left[1 - 6\left(\frac{b}{a}\right)^2 \right] \right\} \right. \\
& \quad \left. - \left\{ 4f_1^2 \left[1 - \left(\frac{b}{a}\right)^2 \right] \left[1 - 3\left(\frac{b}{a}\right)^2 \right] + 24f_1 f_2 \left[1 - \left(\frac{b}{a}\right)^2 \right] \left[1 - 7\left(\frac{b}{a}\right)^2 \right] + 36f_2^2 \left[1 - \left(\frac{b}{a}\right)^2 \right] \left[1 - 11\left(\frac{b}{a}\right)^2 \right] \right\} \right\} \\
& \quad - \left\{ \frac{1}{R^2} \frac{\partial^2}{\partial u^2} \frac{\partial^2}{\partial v^2} - \frac{1}{R^2} \frac{\partial^2}{\partial u^2} \frac{\partial^2}{\partial v^2} \right\} = \frac{1}{R^2} \cos \theta \left\{ 2f_1 \left[1 - 3\left(\frac{b}{a}\right)^2 \right] + 6f_2 \left[1 - \left(\frac{b}{a}\right)^2 \right] \left[1 - 11\left(\frac{b}{a}\right)^2 \right] \right\}
\end{aligned}$$

The terms for the particular integral is then, multiplied by R^2

$$\begin{aligned}
& 2f_1 \left[1 - 2\left(\frac{b}{a}\right)^2 \right] + 6f_2 \left[1 - \left(\frac{b}{a}\right)^2 \right]^4 \left[1 - 6\left(\frac{b}{a}\right)^2 \right] - 4f_1^2 \left[1 - \left(\frac{b}{a}\right)^2 \right] \left[1 - 3\left(\frac{b}{a}\right)^2 \right] \\
& - 24f_1 f_2 \left[1 - \left(\frac{b}{a}\right)^2 \right]^5 \left[1 - 7\left(\frac{b}{a}\right)^2 \right] - 36f_2^2 \left[1 - \left(\frac{b}{a}\right)^2 \right]^9 \left[1 - 11\left(\frac{b}{a}\right)^2 \right] \\
& + \cos \theta \left\{ - 2f_1 \left(\frac{b}{a}\right)^2 - 30f_2 \left[1 - \left(\frac{b}{a}\right)^2 \right]^4 \left(\frac{b}{a}\right)^2 \right\}
\end{aligned}$$

1	$(\frac{a}{a})^2$	$(\frac{a}{a})^4$	$(\frac{a}{a})^6$	$(\frac{a}{a})^8$	$(\frac{a}{a})^{10}$	$(\frac{a}{a})^{12}$	$(\frac{a}{a})^{14}$	$(\frac{a}{a})^{16}$	$(\frac{a}{a})^{18}$	$(\frac{a}{a})^{20}$
$+2f_1$	$-4f_1$									
$+6f_2$	$-24f_2$	$+36f_2$	$-24f_2$	$+6f_2$	$-36f_2$					
	$-36f_2$	$+144f_2$	$-216f_2$	$+144f_2$						
$-4f_1^2$	$+16f_1^2$	$-12f_1^2$								
$-48f_1f_2$	$+240f_1f_2$	$-480f_1f_2$	$+480f_1f_2$	$-240f_1f_2$	$+48f_1f_2$	$-336f_1f_2$	χ	$\frac{1}{2}$		
	$+336f_1f_2$	$-1680f_1f_2$	$+3360f_1f_2$	$-3360f_1f_2$	$+1680f_1f_2$					
$-36f_2^2$	$+324f_2^2$	$-1296f_2^2$	$+3024f_2^2$	$-4536f_2^2$	$+4536f_2^2$	$-3024f_2^2$	$+1296f_2^2$	$-324f_2^2$	$+36f_2^2$	$-396f_2^2$
	$+396f_2^2$	$-3564f_2^2$	$+14256f_2^2$	$-33264f_2^2$	$+49896f_2^2$	$-49896f_2^2$	$+33264f_2^2$	$-14256f_2^2$	$+3564f_2^2$	
$\frac{1}{16 \cdot 4}$	$\frac{1}{36 \cdot 16}$	$\frac{1}{64 \cdot 36}$	$\frac{1}{100 \cdot 64}$	$\frac{1}{144 \cdot 100}$	$\frac{1}{196 \cdot 144}$	$\frac{1}{256 \cdot 196}$	$\frac{1}{324 \cdot 256}$	$\frac{1}{400 \cdot 324}$	$\frac{1}{484 \cdot 400}$	$\frac{1}{576 \cdot 484}$
	$-2f_1$									
	$-30f_2$	$+120f_2$	$-180f_2$	$+120f_2$	$-30f_2$					
	$\frac{1}{32 \cdot 12}$	$\frac{1}{60 \cdot 32}$	$\frac{1}{96 \cdot 60}$	$\frac{1}{140 \cdot 96}$	$\frac{1}{192 \cdot 140}$					

$$\begin{aligned}
\frac{\partial}{\partial \alpha} = E(K) & \left[\frac{1}{4} g \left(\frac{\alpha}{a} \right)^2 + \frac{1}{32} (f_1 + 3f_2 - 2f_1^2 - 12f_1f_2 - 18f_2^2) \left(\frac{\alpha}{a} \right)^4 + \frac{1}{114} (-f_1 - 15f_2 + 4f_1^2 + 72f_1f_2 + 180f_2^2) \left(\frac{\alpha}{a} \right)^6 \right. \\
& + \frac{1}{192} (15f_2 - f_1^2 - 90f_1f_2 - 405f_2^2) \left(\frac{\alpha}{a} \right)^8 + \frac{1}{400} (-15f_2 + 120f_1f_2 + 1080f_2^2) \left(\frac{\alpha}{a} \right)^{10} \\
& + \frac{1}{96} (f_2 - 12f_1f_2 - 1252f_2^2) \left(\frac{\alpha}{a} \right)^{12} + \frac{1}{184} (-f_2 + 24f_1f_2 + 1512f_2^2) \left(\frac{\alpha}{a} \right)^{14} \\
& + \frac{1}{896} (-3ff_2 - 945f_2^2) \left(\frac{\alpha}{a} \right)^{16} + \frac{5}{12} f_2^2 \left(\frac{\alpha}{a} \right)^{18} - \frac{9}{80} f_2^2 \left(\frac{\alpha}{a} \right)^{20} + \frac{9}{484} f_2^2 \left(\frac{\alpha}{a} \right)^{22} - \frac{1}{384} f_2^2 \left(\frac{\alpha}{a} \right)^{24} \\
& + \left. \cos 2\theta \left\{ -\frac{1}{192} (f_1 + 15f_2) \left(\frac{\alpha}{a} \right)^6 + \frac{1}{16} f_2 \left(\frac{\alpha}{a} \right)^8 - \frac{1}{32} f_2 \left(\frac{\alpha}{a} \right)^{10} + \frac{1}{112} f_2 \left(\frac{\alpha}{a} \right)^{12} - \frac{1}{896} f_2 \left(\frac{\alpha}{a} \right)^{14} + \frac{1}{64} f_2 \left(\frac{\alpha}{a} \right)^{16} \right\} \right] \\
\frac{1}{2} \frac{\partial^2}{\partial \alpha^2} = E(K^2) & \left[\frac{1}{240} + \frac{1}{8} A \left(\frac{\alpha}{a} \right)^2 + \frac{1}{24} B \left(\frac{\alpha}{a} \right)^4 + \frac{1}{24} C \left(\frac{\alpha}{a} \right)^6 + \frac{1}{40} D \left(\frac{\alpha}{a} \right)^8 + \frac{1}{8} E \left(\frac{\alpha}{a} \right)^{10} + \frac{1}{56} F \left(\frac{\alpha}{a} \right)^{12} \right. \\
& + \frac{1}{56} G \left(\frac{\alpha}{a} \right)^{14} + \frac{15}{2} H \left(\frac{\alpha}{a} \right)^{16} - \frac{9}{4} I \left(\frac{\alpha}{a} \right)^{18} + \frac{9}{32} J \left(\frac{\alpha}{a} \right)^{20} - \frac{3}{88} K \left(\frac{\alpha}{a} \right)^{22} \\
& + \left. \cos 2\theta \left\{ -\frac{6}{192} (f_1 + 15f_2) \left(\frac{\alpha}{a} \right)^4 + \frac{2}{16} f_2 \left(\frac{\alpha}{a} \right)^6 - \frac{10}{32} f_2 \left(\frac{\alpha}{a} \right)^8 + \frac{12}{112} f_2 \left(\frac{\alpha}{a} \right)^{10} - \frac{14}{896} f_2 \left(\frac{\alpha}{a} \right)^{12} \right. \right. \\
& \quad \left. \left. + 2f_2 + 4f_2 \left(\frac{\alpha}{a} \right)^2 \right\} \right]
\end{aligned}$$

$$\frac{1}{n^2} \frac{\partial^2 \Phi}{\partial \beta^2} = E\left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ \frac{4}{192} (I_1 + 15I_2) \left(\frac{a}{a}\right)^4 - \frac{4}{16} I_2 \left(\frac{a}{a}\right)^6 + \frac{4}{32} I_2 \left(\frac{a}{a}\right)^8 - \frac{4}{112} I_2 \left(\frac{a}{a}\right)^{10} + \frac{4}{896} I_2 \left(\frac{a}{a}\right)^{12} - 4\beta_2 - 4\alpha_2 \left(\frac{a}{a}\right)^2 \right\}$$

$$\hat{n} = E\left(\frac{a}{R}\right)^2 \left[\frac{1}{2} I_0 + \frac{1}{8} A \left(\frac{a}{a}\right)^2 + \frac{1}{24} B \left(\frac{a}{a}\right)^4 + \frac{1}{40} D \left(\frac{a}{a}\right)^6 + \frac{1}{8} E \left(\frac{a}{a}\right)^{10} + \frac{1}{56} F \left(\frac{a}{a}\right)^{12} \right.$$

$$\left. + \frac{1}{56} G \left(\frac{a}{a}\right)^{14} + \frac{15}{2} H \left(\frac{a}{a}\right)^{16} - \frac{9}{4} I \left(\frac{a}{a}\right)^{18} + \frac{9}{32} J \left(\frac{a}{a}\right)^{20} - \frac{3}{88} K \left(\frac{a}{a}\right)^{22} \right.$$

$$\left. + \cos 2\theta \left\{ -\frac{2}{192} (I_1 + 15I_2) \left(\frac{a}{a}\right)^4 + \frac{4}{16} I_2 \left(\frac{a}{a}\right)^6 - \frac{4}{32} I_2 \left(\frac{a}{a}\right)^8 + \frac{4}{112} I_2 \left(\frac{a}{a}\right)^{10} - \frac{4}{896} I_2 \left(\frac{a}{a}\right)^{12} - 2\beta_2 \right\} \right\}$$

$$\hat{\theta} = E\left(\frac{a}{R}\right)^2 \left[\frac{1}{2} I_0 + \frac{3}{8} A \left(\frac{a}{a}\right)^2 + \frac{5}{24} B \left(\frac{a}{a}\right)^4 + \frac{7}{24} C \left(\frac{a}{a}\right)^6 + \frac{9}{40} D \left(\frac{a}{a}\right)^8 + \frac{11}{8} E \left(\frac{a}{a}\right)^{10} + \frac{13}{56} F \left(\frac{a}{a}\right)^{12} \right.$$

$$\left. + \frac{15}{56} G \left(\frac{a}{a}\right)^{14} + \frac{15 \times 17}{2} H \left(\frac{a}{a}\right)^{16} - \frac{9 \times 19}{4} I \left(\frac{a}{a}\right)^{18} + \frac{9 \times 21}{32} J \left(\frac{a}{a}\right)^{20} - \frac{3 \times 23}{88} K \left(\frac{a}{a}\right)^{22} \right.$$

$$\left. + \cos 2\theta \left\{ -\frac{30}{192} (I_1 + 15I_2) \left(\frac{a}{a}\right)^4 + \frac{56}{16} I_2 \left(\frac{a}{a}\right)^6 - \frac{90}{32} I_2 \left(\frac{a}{a}\right)^8 + \frac{132}{112} I_2 \left(\frac{a}{a}\right)^{10} - \frac{162}{896} I_2 \left(\frac{a}{a}\right)^{12} + \frac{12}{896} I_2 \left(\frac{a}{a}\right)^{14} \right\} \right\}$$

$$\hat{n} = E\left(\frac{a}{R}\right)^2 \sin 2\theta \left\{ -\frac{10}{192} (I_1 + 15I_2) \left(\frac{a}{a}\right)^4 + \frac{14}{16} I_2 \left(\frac{a}{a}\right)^6 - \frac{16}{32} I_2 \left(\frac{a}{a}\right)^8 + \frac{22}{112} I_2 \left(\frac{a}{a}\right)^{10} - \frac{26}{896} I_2 \left(\frac{a}{a}\right)^{12} + 2\beta_2 + 6\alpha_2 \left(\frac{a}{a}\right)^2 \right\}$$

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$$\begin{aligned}
\hat{n}_0 - \hat{n}_0 &= E\left(\frac{1}{R}\right) \left[\frac{1}{2} (1-\nu) \rho_0 + \frac{(1-3\nu)}{8} A\left(\frac{1}{a}\right)^2 + \frac{(1-5\nu)}{24} B\left(\frac{1}{a}\right)^4 + \frac{(1-7\nu)}{24} C\left(\frac{1}{a}\right)^6 + \frac{(1-9\nu)}{40} D\left(\frac{1}{a}\right)^8 + \frac{(1-11\nu)}{18} E\left(\frac{1}{a}\right)^{10} \right. \\
&+ \frac{(1-13\nu)}{56} F\left(\frac{1}{a}\right)^{12} \\
&+ \frac{(1-15\nu)}{56} G\left(\frac{1}{a}\right)^{14} + \frac{15(1-17\nu)}{2} H\left(\frac{1}{a}\right)^{16} - \frac{9(1-19\nu)}{4} I\left(\frac{1}{a}\right)^{18} + \frac{9(1-21\nu)}{22} J\left(\frac{1}{a}\right)^{20} - \frac{3(1-23\nu)}{88} K\left(\frac{1}{a}\right)^{22} \\
&+ \cos 2\theta \left\{ -\frac{(1-15\nu)}{96} (\rho_1 + 15\rho_2) \left(\frac{1}{a}\right)^4 + \frac{(1-14\nu)}{4} \rho_2 \left(\frac{1}{a}\right)^6 - \frac{(3-45\nu)}{16} \rho_2 \left(\frac{1}{a}\right)^8 + \frac{(2-33\nu)}{28} \rho_2 \left(\frac{1}{a}\right)^{10} - \frac{(5-91\nu)}{448} \rho_2 \left(\frac{1}{a}\right)^{12} \right. \\
&\quad \left. - 2(1+\nu) \rho_2 - 12\nu \rho_2 \left(\frac{1}{a}\right)^2 \right\} \Bigg]
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2} \left(\frac{\partial u}{\partial r} \right)^2 - \frac{1}{2} \left(\frac{\partial u}{\partial \theta} \right)^2 &= \frac{1}{R^2} \left[-\rho_1 \left[1 - \left(\frac{1}{a}\right)^2 \right] \left(\frac{1}{a}\right)^2 - 3\rho_2 \left[1 - \left(\frac{1}{a}\right)^2 \right] \left(\frac{1}{a}\right)^4 + 2\rho_1^2 \left[1 - \left(\frac{1}{a}\right)^2 \right] \left(\frac{1}{a}\right)^6 \right. \\
&+ 18\rho_2^2 \left[1 - \left(\frac{1}{a}\right)^2 \right] \left(\frac{1}{a}\right)^{10} + \cos 2\theta \left\{ \rho_1 \left[1 - \left(\frac{1}{a}\right)^2 \right] \left(\frac{1}{a}\right)^2 + 3\rho_2 \left[1 - \left(\frac{1}{a}\right)^2 \right] \left(\frac{1}{a}\right)^4 \right\} \Bigg] \\
&= \frac{1}{R^2} \left[-A\left(\frac{1}{a}\right)^2 - 8B\left(\frac{1}{a}\right)^4 - 2C\left(\frac{1}{a}\right)^6 - 2D\left(\frac{1}{a}\right)^8 - 15E\left(\frac{1}{a}\right)^{10} - 3F\left(\frac{1}{a}\right)^{12} - 4G\left(\frac{1}{a}\right)^{14} - 216H\left(\frac{1}{a}\right)^{16} \right. \\
&+ 810I\left(\frac{1}{a}\right)^{18} - 180J\left(\frac{1}{a}\right)^{20} + 18K\left(\frac{1}{a}\right)^{22} \\
&+ \cos 2\theta \left\{ (\rho_1 + 3\rho_2) \left(\frac{1}{a}\right)^2 - (\rho_1 + 15\rho_2) \left(\frac{1}{a}\right)^4 + 30\rho_2 \left(\frac{1}{a}\right)^6 - 30\rho_2 \left(\frac{1}{a}\right)^8 + 15\rho_2 \left(\frac{1}{a}\right)^{10} - 3\rho_2 \left(\frac{1}{a}\right)^{12} \right\} \Bigg]
\end{aligned}$$

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$\left(\frac{A}{a}\right)^2$	$\left(\frac{A}{a}\right)^4$	$\left(\frac{A}{a}\right)^6$	$\left(\frac{A}{a}\right)^8$	$\left(\frac{A}{a}\right)^{10}$	$\left(\frac{A}{a}\right)^{12}$	$\left(\frac{A}{a}\right)^{14}$	$\left(\frac{A}{a}\right)^{16}$	$\left(\frac{A}{a}\right)^{18}$	$\left(\frac{A}{a}\right)^{20}$	$\left(\frac{A}{a}\right)^{22}$
$-f_1$	$+f_1$									
$-3f_2$	$+15f_2$	$-30f_2$	$+30f_2$	$-15f_2$	$+3f_2$					
$+2f_1^2$	$-4f_1^2$	$+2f_1^2$								
$+12f_1f_2$	$-22f_1f_2$	$+18f_1f_2$	$-240f_1f_2$	$+160f_1f_2$	$-72f_1f_2$	$+12f_1f_2$				
$+18f_2^2$	$-180f_2^2$	$+810f_2^2$	$-2160f_2^2$	$+3240f_2^2$	$-4536f_2^2$	$+3240f_2^2$	$-2160f_2^2$	$+810f_2^2$	$-180f_2^2$	$+18f_2^2$
$+f_1$	$-f_1$									
$+3f_2$	$-15f_2$	$+30f_2$	$-30f_2$	$+15f_2$	$-3f_2$					

$$\begin{aligned}
\frac{2H}{\partial r} = & \left(\frac{a}{R} \right)^2 \left[\frac{1}{2} (1-v) q_0 + \frac{3(3-v)}{8} A \left(\frac{a}{a} \right)^2 + \frac{5(5-v)}{24} B \left(\frac{a}{a} \right)^4 + \frac{7(7-v)}{24} C \left(\frac{a}{a} \right)^6 + \frac{9(9-v)}{40} D \left(\frac{a}{a} \right)^8 \right. \\
& + \frac{11(11-v)}{8} E \left(\frac{a}{a} \right)^{10} + \frac{13(13-v)}{56} F \left(\frac{a}{a} \right)^{12} + \frac{15(15-v)}{56} G \left(\frac{a}{a} \right)^{14} + \frac{15 \times 12(17-v)}{2} H \left(\frac{a}{a} \right)^{16} - \frac{9 \times 19(19-v)}{4} I \left(\frac{a}{a} \right)^{18} \\
& + \frac{9 \times 21(21-v)}{22} J \left(\frac{a}{a} \right)^{20} - \frac{3 \times 23(23-v)}{88} K \left(\frac{a}{a} \right)^{22} \\
& + \cos 2\theta \left\{ - (j_1' + 3j_2') \left(\frac{a}{a} \right)^2 + \frac{(95+15v)}{96} (j_1' + 15j_2') \left(\frac{a}{a} \right)^4 - \frac{(119+14v)}{4} j_2' \left(\frac{a}{a} \right)^6 + \frac{(477+45v)}{16} j_2' \left(\frac{a}{a} \right)^8 \right. \\
& \left. - \frac{(418+33v)}{24} j_2' \left(\frac{a}{a} \right)^{10} + \frac{(1339+91v)}{448} j_2' \left(\frac{a}{a} \right)^{12} - 2(1+v) j_2' - 12v j_2' \left(\frac{a}{a} \right)^2 \right\}
\end{aligned}$$

$$\begin{aligned}
\frac{H}{R} = & \left(\frac{a}{R} \right)^3 \left[\frac{1}{2} (1-v) q_0 \left(\frac{a}{a} \right) + \frac{(3-v)}{8} A \left(\frac{a}{a} \right)^3 + \frac{(5-v)}{24} B \left(\frac{a}{a} \right)^5 + \frac{(7-v)}{24} C \left(\frac{a}{a} \right)^7 + \frac{(9-v)}{40} D \left(\frac{a}{a} \right)^9 \right. \\
& + \frac{(11-v)}{8} E \left(\frac{a}{a} \right)^{11} + \frac{(13-v)}{56} F \left(\frac{a}{a} \right)^{13} + \frac{(15-v)}{56} G \left(\frac{a}{a} \right)^{15} + \frac{15(17-v)}{2} H \left(\frac{a}{a} \right)^{17} - \frac{9(19-v)}{4} I \left(\frac{a}{a} \right)^{19} \\
& + \frac{9(21-v)}{22} J \left(\frac{a}{a} \right)^{21} - \frac{3(23-v)}{88} K \left(\frac{a}{a} \right)^{23} \\
& + \cos 2\theta \left\{ -\frac{4}{3} (j_1' + 3j_2') \left(\frac{a}{a} \right)^3 + \frac{(19+3v)}{96} (j_1' + 15j_2') \left(\frac{a}{a} \right)^5 - \frac{(17+2v)}{4} j_2' \left(\frac{a}{a} \right)^7 + \frac{(53+5v)}{16} j_2' \left(\frac{a}{a} \right)^9 \right. \\
& \left. - \frac{(38+3v)}{24} j_2' \left(\frac{a}{a} \right)^{11} + \frac{(103+7v)}{448} j_2' \left(\frac{a}{a} \right)^{13} - 2(1+v) j_2' \left(\frac{a}{a} \right) - 4v j_2' \left(\frac{a}{a} \right)^3 \right\}
\end{aligned}$$

The non-uniform part of $\partial\bar{\partial} - \gamma\bar{\gamma}\partial\bar{\partial}$ is

$$E\left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ -\frac{(15-v)}{96} (f_1 + 15f_2) \left(\frac{a}{a}\right)^4 + \frac{(14-v)}{4} \rho_2 \left(\frac{a}{a}\right)^6 - \frac{(45-3v)}{16} \rho_2 \left(\frac{a}{a}\right)^8 + \frac{(33-2v)}{28} \rho_2 \left(\frac{a}{a}\right)^{10} - \frac{(91-5v)}{488} \rho_2 \left(\frac{a}{a}\right)^{12} + 2(14v) \rho_2 + 12a_2 \left(\frac{a}{a}\right)^2 \right\}$$

$$\frac{1}{a} \frac{\partial^2 \psi}{\partial \theta^2} = \left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ \frac{1}{3} (f_1 + 3f_2) \left(\frac{a}{a}\right)^2 - \frac{(34+2v)}{96} (f_1 + 15f_2) \left(\frac{a}{a}\right)^4 + \frac{(31+v)}{4} \rho_2 \left(\frac{a}{a}\right)^6 - \frac{(98+2v)}{16} \rho_2 \left(\frac{a}{a}\right)^8 + \frac{(71+v)}{24} \rho_2 \left(\frac{a}{a}\right)^{10} - \frac{(194+2v)}{448} \rho_2 \left(\frac{a}{a}\right)^{12} + 4(14v) \rho_2 + 4(3+4v) \left(\frac{a}{a}\right)^2 \right\}$$

$$\frac{\psi}{R} = \left(\frac{a}{R}\right)^3 \sin 2\theta \left\{ \frac{1}{6} (f_1 + 3f_2) \left(\frac{a}{a}\right)^3 - \frac{(17+v)}{96} (f_1 + 15f_2) \left(\frac{a}{a}\right)^5 + \frac{(31+v)}{8} \rho_2 \left(\frac{a}{a}\right)^7 - \frac{(49+v)}{16} \rho_2 \left(\frac{a}{a}\right)^9 + \frac{(71+v)}{56} \rho_2 \left(\frac{a}{a}\right)^{11} - \frac{(97+v)}{448} \rho_2 \left(\frac{a}{a}\right)^{13} + 2(14v) \rho_2 + 2(3+4v) \left(\frac{a}{a}\right)^3 \right\}$$

Now the non-uniform parts

$$\hat{u}_a = \cos 2\theta \left\{ -\frac{1}{96} p_1 - \frac{15}{448} p_2 - \frac{1}{2} p_3 \right\} - \left\{ \frac{1}{2} - 6p_2 - 4s_2 \right\}$$

$$\hat{\theta}_a = \cos 2\theta \left\{ -\frac{5}{32} p_1 - \frac{305}{448} p_2 + \frac{1}{2} p_3 + 12s_2 \right\} - \left\{ 6p_2 - \frac{1}{2} \right\}$$

$$\hat{v}_a = \sin 2\theta \left\{ -\frac{5}{96} p_1 - \frac{135}{448} p_2 + \frac{1}{2} p_3 + 6s_2 \right\} - \left\{ -\frac{1}{2} - 6p_2 - 2s_2 \right\}$$

$$\left(\frac{u}{R} \right)_a = \cos 2\theta \left\{ -\frac{(13-3v)}{96} p_1 - \frac{43-85v}{448} p_2 - 2(1+v)p_3 - 4s_2 \right\} - \left\{ \frac{1}{2}(11v) + 2(1+v)p_2 + 4s_2 \right\}$$

$$\left(\frac{v}{R} \right)_a = \sin 2\theta \left\{ -\frac{(1+v)}{96} p_1 - \frac{131+35v}{448} p_2 + 2(1+v)p_3 + 2(3+v)s_2 \right\} - \left\{ 2(1+v)p_2 - \frac{1}{2}(1+v) \right\}$$

$$\frac{141.5}{114.8}$$

$$\frac{14.5}{114.8}$$

$$\begin{aligned}
f_2 + 0.6666667 S_2 - 0.08333333 &= \eta \{ +0.33333333 p_2 + 0 + 0.00173611 f_1 + 0.00551036 f_2 \} \\
f_2 + 0 - 0.08333333 &= \eta \{ +0.33333333 p_2 + 2n_2 - 0.02604167 f_1 - 0.11346726 f_2 \} \\
f_2 + 0.3333333 S_2 + 0.08333333 &= \eta \{ -0.33333333 p_2 - n_2 + 0.00868056 f_1 + 0.05022321 f_2 \} \\
f_2 + 1.53846154 S_2 + 0.25000 &= \eta \{ -p_2 - 0.76153846 n_2 - 0.0444756 f_1 - 0.01502404 f_2 \} \\
f_2 + 0 - 0.25000 &= \eta \{ +f_2 + 2.53846154 n_2 - 0.00520833 f_1 - 0.12148008 f_2 \}
\end{aligned}$$

$$\begin{aligned}
0.6666667 S_2 + 0 &= \eta \{ 0 - 2n_2 + 0.02777778 f_1 + 0.11904762 f_2 \} \\
0.3333333 S_2 + 0.6666667 &= \eta \{ -0.6666667 p_2 - 3n_2 + 0.03472222 f_1 + 0.16369047 f_2 \} \\
1.2057621 S_2 + 0.1666667 &= \eta \{ -0.6666667 p_2 + 0.53846154 f_1 - 0.05715812 f_1 - 0.0652425 f_2 \} \\
1.53846154 S_2 + 0.50000 &= \eta \{ -2p_2 - 3n_2 - 0.04326923 f_1 + 0.10645604 f_2 \}
\end{aligned}$$

$$\begin{aligned}
S_2 + 0 &= \eta \{ 0 - 3n_2 + 0.04166667 f_1 + 0.17857143 f_2 \} \\
S_2 + 0.50000 &= \eta \{ -2p_2 - 9n_2 + 0.10416667 f_1 + 0.49107141 f_2 \} \\
S_2 + 0.13829787 &= \eta \{ -0.55319149 p_2 + 0.44680851 n_2 - 0.04742908 f_1 - 0.05414133 f_2 \} \\
S_2 + 0.325000 &= \eta \{ -1.3 p_2 - 1.95 n_2 - 0.028125000 f_1 + 0.06917643 f_2 \}
\end{aligned}$$

$$\begin{aligned}
0.50000000 &= \eta \{ -2p_2 - 6n_2 + 0.06250000 f_1 + 0.31250000 f_2 \} \\
0.36170213 &= \eta \{ -1.44680851 p_2 - 9.44680851 n_2 + 0.15159575 f_1 + 0.54521274 f_2 \} \\
0.18670213 &= \eta \{ -0.74680851 p_2 - 2.39680851 n_2 + 0.01930408 f_1 + 0.12333771 f_2 \}
\end{aligned}$$

$$0.25000000 = \gamma \left\{ -f_2 - 3n_2 + 0.03125000f_1 + 0.15625000f_2 \right\}$$

$$0.25000000 = \gamma \left\{ -f_2 - 6.52941176n_2 + 0.10477942f_1 + 0.37683822f_2 \right\}$$

$$0.25000000 = \gamma \left\{ -f_2 - 3.20940171n_2 + 0.02584877f_1 + 0.16515313f_2 \right\}$$

$$3.52941176n_2 = 0.07352942f_1 + 0.22058822f_2$$

$$3.32001005n_2 = 0.07893065f_1 + 0.21168509f_2$$

$$3n_2 = 0.06250000f_1 + 0.18750000f_2$$

$$3n_2 = 0.07132266f_1 + 0.19128113f_2$$

$$0 = 0.00882266f_1 + 0.00378113f_2$$

New Calculation

for the circular region,

$$\frac{w_0}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left\{ 1 - \left(\frac{a}{a} \right)^2 \sin^2 \theta \right\}$$

$$\frac{w}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left\{ 1 - \left(\frac{a}{a} \right)^2 \sin^2 \theta - \frac{1}{4} \left[1 - \left(\frac{a}{a} \right)^2 \right]^2 - \frac{1}{2} \left[1 - \left(\frac{a}{a} \right)^2 \right]^4 \right\}$$

$$\frac{1}{R} \frac{\partial w}{\partial R} = \frac{1}{R} \left\{ -\sin^2 \theta + 2 \frac{1}{4} \left[1 - \left(\frac{a}{a} \right)^2 \right] + 4 \frac{1}{2} \left[1 - \left(\frac{a}{a} \right)^2 \right]^3 \right\}$$

$$\frac{1}{R} \frac{\partial w_0}{\partial R} = \frac{1}{R} \left\{ -\sin^2 \theta \right\}$$

$$\frac{\partial^2 w}{\partial R^2} = \frac{1}{R} \left\{ -\sin^2 \theta + 2 \frac{1}{4} \left[1 - 3 \left(\frac{a}{a} \right)^2 \right] + 4 \frac{1}{2} \left[1 - \left(\frac{a}{a} \right)^2 \right]^2 \left[1 - 7 \left(\frac{a}{a} \right)^2 \right] \right\}$$

$$\frac{\partial^2 w_0}{\partial R^2} = \frac{1}{R} \left\{ -\sin^2 \theta \right\}.$$

$$\frac{1}{R^2} \frac{\partial^2 w}{\partial \theta^2} = \frac{1}{R^2} \frac{\partial^2 w_0}{\partial \theta^2} = \frac{1}{R} \left\{ -\cos \theta \right\}$$

$$\begin{aligned}
& - \left\{ \frac{1}{a} \frac{\partial \omega}{\partial a} \frac{\partial^2 \omega}{\partial a^2} - \frac{1}{a} \frac{\partial \omega}{\partial a} \frac{\partial^2 \omega}{\partial a^2} \right\} \\
& = \frac{1}{R^2} \left\{ (1 - \cos 2\theta) \left\{ 2f_1 \left[1 - 2 \left(\frac{a}{a} \right)^2 \right] + 4f_2 \left[1 - \left(\frac{a}{a} \right)^2 \right] \left[1 - 4 \left(\frac{a}{a} \right)^2 \right] \right\} \right. \\
& \quad \left. - \left\{ 4f_1^2 \left[1 - \left(\frac{a}{a} \right)^2 \right] \left[1 - 3 \left(\frac{a}{a} \right)^2 \right] + 8f_1 f_2 \left[1 - \left(\frac{a}{a} \right)^2 \right] \left[2 - 10 \left(\frac{a}{a} \right)^2 \right] + 16f_2^2 \left[1 - \left(\frac{a}{a} \right)^2 \right]^2 \left[1 - 7 \left(\frac{a}{a} \right)^2 \right] \right\} \right\} \\
& \quad \left. - \left\{ \frac{1}{a^2} \frac{\partial^2 \omega}{\partial a^2} \frac{\partial \omega}{\partial a} - \frac{1}{a^2} \frac{\partial^2 \omega}{\partial a^2} \frac{\partial \omega}{\partial a} \right\} = \frac{1}{R^2} \cos 2\theta \left\{ 2f_1 \left[1 - 3 \left(\frac{a}{a} \right)^2 \right] + 4f_2 \left[1 - \left(\frac{a}{a} \right)^2 \right] \left[1 - 7 \left(\frac{a}{a} \right)^2 \right] \right\} \right\}
\end{aligned}$$

The terms for the particular integral is then, multiplied by R^2

$$\begin{aligned}
& 2f_1 \left[1 - 2 \left(\frac{a}{a} \right)^2 \right] + 4f_2 \left[1 - \left(\frac{a}{a} \right)^2 \right]^2 \left[1 - 4 \left(\frac{a}{a} \right)^2 \right] - 4f_1^2 \left[1 - \left(\frac{a}{a} \right)^2 \right] \left[1 - 3 \left(\frac{a}{a} \right)^2 \right] - 16f_1 f_2 \left[1 - \left(\frac{a}{a} \right)^2 \right]^3 \left[1 - 5 \left(\frac{a}{a} \right)^2 \right] \\
& - 16f_2^2 \left[1 - \left(\frac{a}{a} \right)^2 \right]^5 \left[1 - 7 \left(\frac{a}{a} \right)^2 \right] \\
& + \cos 2\theta \left\{ -2f_1 \left(\frac{a}{a} \right)^2 - 12f_2 \left[1 - \left(\frac{a}{a} \right)^2 \right] \left(\frac{a}{a} \right)^2 \right\}
\end{aligned}$$

1	$\left(\frac{a}{n}\right)^2$	$\left(\frac{a}{n}\right)^4$	$\left(\frac{a}{n}\right)^6$	$\left(\frac{a}{n}\right)^8$	$\left(\frac{a}{n}\right)^{10}$	$\left(\frac{a}{n}\right)^{12}$
$+2f_1$	$-4f_1$					
$+4f_2$	$-8f_2$	$+4f_2$				
	$-16f_2$	$+32f_2$	$-16f_2$			
$-4f_1^2$	$+16f_1^2$	$-12f_1^2$				
$-16f_1f_2$	$+48f_1f_2$	$-48f_1f_2$	$+16f_1f_2$			
	$+80f_1f_2$	$-240f_1f_2$	$+240f_1f_2$	$+80f_1f_2$		
$-16f_2^2$	$+80f_2^2$	$-160f_2^2$	$+160f_2^2$	$-80f_2^2$	$+16f_2^2$	
	$+112f_2^2$	$-560f_2^2$	$+1120f_2^2$	$-1120f_2^2$	$+560f_2^2$	$-112f_2^2$
$\frac{1}{16.4}$	$\frac{1}{36.16}$	$\frac{1}{64.36}$	$\frac{1}{100.64}$	$\frac{1}{144.100}$	$\frac{1}{196.144}$	$\frac{1}{256.196}$

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$$\begin{aligned}
\frac{\Phi}{R^2} &= E\left(\frac{a}{R}\right)^4 \left[\frac{1}{4} \frac{a}{R_0} \left(\frac{a}{a}\right)^2 + \frac{1}{32} (1 + 2\frac{a}{a_2} - 1\frac{a}{a_1} - 8\frac{a}{a_1 a_2} - 8\frac{a}{a_2^2}) \left(\frac{a}{a}\right)^4 + \frac{1}{144} (-1\frac{a}{a_1} - 6\frac{a}{a_2} + 4\frac{a}{a_1^2} + 32\frac{a}{a_1 a_2} + 48\frac{a}{a_2^2}) \left(\frac{a}{a}\right)^6 \right. \\
&+ \frac{1}{192} (3\frac{a}{a_2} - 1\frac{a}{a_1} - 24\frac{a}{a_1 a_2} - 60\frac{a}{a_2^2}) \left(\frac{a}{a}\right)^8 + \frac{1}{400} (-\frac{a}{a_2} + 16\frac{a}{a_1 a_2} + 80\frac{a}{a_2^2}) \left(\frac{a}{a}\right)^{10} \\
&+ \frac{1}{180} (1\frac{a}{a_1 a_2} - 15\frac{a}{a_2^2}) \left(\frac{a}{a}\right)^{12} + \frac{1}{49} \frac{a}{a_2} \left(\frac{a}{a}\right)^{14} - \frac{1}{448} \frac{a}{a_2} \left(\frac{a}{a}\right)^{16} \\
&\left. + \cos 2\theta \left\{ -\frac{1}{192} (1 + 6\frac{a}{a_2}) \left(\frac{a}{a}\right)^6 + \frac{1}{80} \frac{a}{a_2} \left(\frac{a}{a}\right)^8 - \frac{1}{480} \frac{a}{a_2} \left(\frac{a}{a}\right)^{10} + \frac{1}{a_2} \left(\frac{a}{a}\right)^{12} + \frac{1}{a_2} \left(\frac{a}{a}\right)^{14} \right\} \right]
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2} \frac{\partial \Phi}{\partial R} &= E\left(\frac{a}{R}\right)^3 \left[\frac{1}{2} \frac{a}{R_0} + \frac{1}{8} (1 + 2\frac{a}{a_2} - 2\frac{a}{a_1} - 8\frac{a}{a_1 a_2} - 8\frac{a}{a_2^2}) \left(\frac{a}{a}\right)^2 + \frac{1}{24} (-1\frac{a}{a_1} - 6\frac{a}{a_2} + 4\frac{a}{a_1^2} + 32\frac{a}{a_1 a_2} + 48\frac{a}{a_2^2}) \left(\frac{a}{a}\right)^4 \right. \\
&+ \frac{1}{24} (3\frac{a}{a_2} - 1\frac{a}{a_1} - 24\frac{a}{a_1 a_2} - 60\frac{a}{a_2^2}) \left(\frac{a}{a}\right)^6 + \frac{1}{40} (-\frac{a}{a_2} + 16\frac{a}{a_1 a_2} + 80\frac{a}{a_2^2}) \left(\frac{a}{a}\right)^8 + \frac{1}{15} (1\frac{a}{a_1 a_2} - 15\frac{a}{a_2^2}) \left(\frac{a}{a}\right)^{10} \\
&\left. + \frac{2}{3} \frac{a}{a_2} \left(\frac{a}{a}\right)^{12} - \frac{1}{28} \frac{a}{a_2} \left(\frac{a}{a}\right)^{14} + \cos 2\theta \left\{ -\frac{1}{192} (1 + 6\frac{a}{a_2}) \left(\frac{a}{a}\right)^4 + \frac{1}{80} \frac{a}{a_2} \left(\frac{a}{a}\right)^6 - \frac{1}{480} \frac{a}{a_2} \left(\frac{a}{a}\right)^8 \right. \right. \\
&\left. \left. + 2\frac{a}{a_2} + 4\frac{a}{a_2} \left(\frac{a}{a}\right)^2 \right\} \right]
\end{aligned}$$

$$\frac{1}{R^2} \frac{\partial \Phi}{\partial R^2} = E\left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ + \frac{1}{192} (1 + 6\frac{a}{a_2}) \left(\frac{a}{a}\right)^4 - \frac{1}{80} \frac{a}{a_2} \left(\frac{a}{a}\right)^6 + \frac{1}{480} \frac{a}{a_2} \left(\frac{a}{a}\right)^8 - 4\frac{a}{a_2} - 4\frac{a}{a_2} \left(\frac{a}{a}\right)^4 \right\}$$

$$\begin{aligned} \tilde{N} = E\left(\frac{a}{R}\right)^2 & \left[\frac{1}{2} a_0 + \frac{1}{8} (a_1 + 2a_2 - 8a_1^2 - 8a_1a_2 - 8a_2^2) \left(\frac{a}{a}\right)^2 + \frac{1}{24} (-a_1 - 6a_2 + 4a_1^2 + 32a_1a_2 + 48a_2^2) \left(\frac{a}{a}\right)^4 \right. \\ & + \frac{1}{24} (3a_2 - a_1^2 - 24a_1a_2 - 60a_2^2) \left(\frac{a}{a}\right)^6 + \frac{1}{40} (-a_2 + 16a_1a_2 + 80a_2^2) \left(\frac{a}{a}\right)^8 + \frac{1}{15} (a_1^2 - 15a_2^2) \left(\frac{a}{a}\right)^{10} \\ & \left. + \frac{2}{7} a_2^2 \left(\frac{a}{a}\right)^{12} - \frac{1}{28} a_2^2 \left(\frac{a}{a}\right)^{14} + \cos 2\theta \left\{ -\frac{1}{96} (a_1 + 6a_2) \left(\frac{a}{a}\right)^4 + \frac{1}{20} a_2 \left(\frac{a}{a}\right)^6 - \frac{1}{80} a_2 \left(\frac{a}{a}\right)^{10} - 2a_2 \right\} \right] \end{aligned}$$

$$\begin{aligned} \tilde{N} = E\left(\frac{a}{R}\right)^2 & \left[\frac{1}{2} a_0 + \frac{3}{8} (a_1 + 2a_2 - 2a_1^2 - 8a_1a_2 - 8a_2^2) \left(\frac{a}{a}\right)^2 + \frac{5}{24} (-a_1 - 6a_2 + 4a_1^2 + 32a_1a_2 + 48a_2^2) \left(\frac{a}{a}\right)^4 \right. \\ & + \frac{7}{24} (3a_2 - a_1^2 - 24a_1a_2 - 60a_2^2) \left(\frac{a}{a}\right)^6 + \frac{9}{40} (-a_2 + 16a_1a_2 + 80a_2^2) \left(\frac{a}{a}\right)^8 + \frac{11}{15} (a_1^2 - 15a_2^2) \left(\frac{a}{a}\right)^{10} \\ & \left. + \frac{26}{7} a_2^2 \left(\frac{a}{a}\right)^{12} - \frac{15}{28} a_2^2 \left(\frac{a}{a}\right)^{14} + \cos 2\theta \left\{ -\frac{15}{96} (a_1 + 6a_2) \left(\frac{a}{a}\right)^4 + \frac{14}{20} a_2 \left(\frac{a}{a}\right)^6 - \frac{15}{80} a_2 \left(\frac{a}{a}\right)^{10} \right. \right. \\ & \left. \left. + 2a_2 + 12a_2 \left(\frac{a}{a}\right)^2 \right\} \right] \end{aligned}$$

$$\tilde{N} = E\left(\frac{a}{R}\right)^2 \sin 2\theta \left\{ -\frac{5}{96} (a_1 + 6a_2) \left(\frac{a}{a}\right)^4 + \frac{7}{40} a_2 \left(\frac{a}{a}\right)^6 - \frac{3}{80} a_2 \left(\frac{a}{a}\right)^{10} + 2a_2 + 6a_2 \left(\frac{a}{a}\right)^2 \right\}$$

$$\begin{aligned}
\Lambda_b - \bar{0} &= E\left(\frac{a}{R}\right)^2 \left[\frac{4}{2} (1-\nu) \rho_0 + \frac{(1-3\nu)}{8} (\rho_1 + 2\rho_2 - 2\rho_1^2 - 8\rho_1\rho_2 - 8\rho_2^2) \left(\frac{a}{a}\right)^2 + \frac{(1-5\nu)}{24} (-\rho_1 - 6\rho_2 + 4\rho_1^2 + 32\rho_1\rho_2 + 48\rho_2^2) \left(\frac{a}{a}\right)^3 \right. \\
&\quad + \frac{(1-7\nu)}{24} (3\rho_2 - \rho_2^3 - 24\rho_1\rho_2 - 60\rho_2^2) \left(\frac{a}{a}\right)^6 + \frac{(1-9\nu)}{40} (-\rho_2 + 16\rho_1\rho_2 + 60\rho_2^2) \left(\frac{a}{a}\right)^8 + \frac{(1-11\nu)}{15} (-\rho_1\rho_2 - 15\rho_2^2) \left(\frac{a}{a}\right)^{10} \\
&\quad + \frac{2(1-13\nu)}{7} \rho_2^2 \left(\frac{a}{a}\right)^{12} - \frac{(1-15\nu)}{28} \rho_2^2 \left(\frac{a}{a}\right)^{14} + c_{020} \left\{ -\frac{96}{7} (\rho_1 + 6\rho_2) \left(\frac{a}{a}\right)^4 + \frac{(1-14\nu)}{20} \rho_2^6 - \frac{(1-15\nu)}{80} \rho_2^2 \left(\frac{a}{a}\right)^8 \right. \\
&\quad \left. \left. - 2(1+\nu)\rho_2 - 12\nu\rho_2 \left(\frac{a}{a}\right)^2 \right\} \right]
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2} \left(\frac{\partial u}{\partial a} \right)^2 - \frac{1}{2} \left(\frac{\partial u}{\partial a} \right)^2 &= \frac{1}{2} \left(\frac{a}{R} \right)^2 \left[(c_{020} - 1) \left\{ 2\rho_1 \left[1 - \left(\frac{a}{a}\right)^2 \right] + 4\rho_2 \left[1 - \left(\frac{a}{a}\right)^2 \right]^3 \right\} \right. \\
&\quad \left. + 4\rho_1^2 \left[1 - \left(\frac{a}{a}\right)^2 \right]^2 + 16\rho_1\rho_2 \left[1 - \left(\frac{a}{a}\right)^2 \right]^4 + 16\rho_2^2 \left[1 - \left(\frac{a}{a}\right)^2 \right]^6 \right] \left(\frac{a}{a}\right)^2 \\
&= \left(\frac{a}{R}\right)^2 \left[-\rho_1 \left[1 - \left(\frac{a}{a}\right)^2 \right] \left(\frac{a}{a}\right)^2 - 2\rho_2 \left[1 - \left(\frac{a}{a}\right)^2 \right] \left(\frac{a}{a}\right)^2 + 2\rho_1^2 \left[1 - \left(\frac{a}{a}\right)^2 \right] \left(\frac{a}{a}\right)^2 + 8\rho_1\rho_2 \left[1 - \left(\frac{a}{a}\right)^2 \right]^4 + 8\rho_2^2 \left[1 - \left(\frac{a}{a}\right)^2 \right]^6 \right. \\
&\quad \left. + c_{020} \left\{ \rho_1 \left[1 - \left(\frac{a}{a}\right)^2 \right] \left(\frac{a}{a}\right)^2 + 2\rho_2 \left[1 - \left(\frac{a}{a}\right)^2 \right]^3 \right\} \right] \\
&= \left(\frac{a}{R}\right)^2 \left[-(\rho_1 + 2\rho_2 - 2\rho_1^2 - 8\rho_1\rho_2 - 8\rho_2^2) \left(\frac{a}{a}\right)^2 - (-\rho_1 - 6\rho_2 + 4\rho_1^2 + 32\rho_1\rho_2 + 48\rho_2^2) \left(\frac{a}{a}\right)^4 \right. \\
&\quad \left. - 2(3\rho_2 - \rho_2^3 - 24\rho_1\rho_2 - 60\rho_2^2) \left(\frac{a}{a}\right)^6 - 9(-\rho_2 + 16\rho_1\rho_2 + 60\rho_2^2) \left(\frac{a}{a}\right)^8 - 8(\rho_1\rho_2 - 15\rho_2^2) \left(\frac{a}{a}\right)^{10} \right. \\
&\quad \left. - 48\rho_2^2 \left(\frac{a}{a}\right)^{12} + 8\rho_2^2 \left(\frac{a}{a}\right)^{14} + c_{020} \left\{ (\rho_1 + 2\rho_2) \left(\frac{a}{a}\right)^2 - (\rho_1 + 6\rho_2) \left(\frac{a}{a}\right)^4 + 66\rho_2^2 \left(\frac{a}{a}\right)^6 - 2\rho_2^2 \left(\frac{a}{a}\right)^8 \right\} \right]
\end{aligned}$$

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$$\begin{aligned}
\frac{\partial u}{\partial t} = & \frac{1}{2} \left[\frac{1}{2} (1-v) \rho_0 + \frac{3(3-v)}{8} (\rho_1 + 2\rho_2 - 2\rho_1^2 - 8\rho_1\rho_2 - 8\rho_2^2) \left(\frac{a}{a_0}\right)^2 + \frac{5(5-v)}{24} (-\rho_1 - 6\rho_2 + 4\rho_1^2 + 32\rho_1\rho_2 + 48\rho_2^2) \left(\frac{a}{a_0}\right)^3 \right. \\
& + \frac{7(7-v)}{24} (3\rho_2 - \rho_2^2 - 24\rho_1\rho_2 - 60\rho_2^2) \left(\frac{a}{a_0}\right)^4 + \frac{9(9-v)}{40} (-\rho_2 + 16\rho_1\rho_2 + 80\rho_2^2) \left(\frac{a}{a_0}\right)^5 + \frac{11(11-v)}{15} (\rho_1\rho_2 - 15\rho_2^2) \left(\frac{a}{a_0}\right)^6 \\
& + 26 \frac{(13-v)}{7} \rho_2^2 \left(\frac{a}{a_0}\right)^{12} - \frac{15(15-v)}{28} \rho_2^2 \left(\frac{a}{a_0}\right)^{14} + \cos \theta \left\{ -(\rho_1 + 2\rho_2) \left(\frac{a}{a_0}\right)^2 + \frac{(95+15v)}{96} (\rho_1 + 6\rho_2) \left(\frac{a}{a_0}\right)^4 - \frac{(119+14v)}{20} \rho_2 \left(\frac{a}{a_0}\right)^6 \right. \\
& \left. + \frac{(159+15v)}{80} \rho_2 \left(\frac{a}{a_0}\right)^8 - 2(1+v) \rho_2^2 - 12v \rho_2 \left(\frac{a}{a_0}\right)^2 \right\} \Bigg]
\end{aligned}$$

$$\begin{aligned}
\frac{u}{a} = & \frac{1}{2} \left[\frac{1}{2} (1-v) \rho_0 \left(\frac{a}{a_0}\right) + \frac{(3-v)}{8} (\rho_1 + 2\rho_2 - 2\rho_1^2 - 8\rho_1\rho_2 - 8\rho_2^2) \left(\frac{a}{a_0}\right)^3 + \frac{(5-v)}{24} (-\rho_1 - 6\rho_2 + 4\rho_1^2 + 32\rho_1\rho_2 + 48\rho_2^2) \left(\frac{a}{a_0}\right)^5 \right. \\
& + \frac{(7-v)}{24} (3\rho_2 - \rho_2^2 - 24\rho_1\rho_2 - 60\rho_2^2) \left(\frac{a}{a_0}\right)^7 + \frac{(9-v)}{40} (-\rho_2 + 16\rho_1\rho_2 + 80\rho_2^2) \left(\frac{a}{a_0}\right)^9 + \frac{(11-v)}{15} (\rho_1\rho_2 - 15\rho_2^2) \left(\frac{a}{a_0}\right)^{11} \\
& + \frac{2(13-v)}{7} \rho_2^2 \left(\frac{a}{a_0}\right)^{12} - \frac{(15-v)}{28} \rho_2^2 \left(\frac{a}{a_0}\right)^{14} + \cos \theta \left\{ -\frac{1}{3} (\rho_1 + 2\rho_2) \left(\frac{a}{a_0}\right)^3 + \frac{(19+3v)}{96} (\rho_1 + 6\rho_2) \left(\frac{a}{a_0}\right)^5 - \frac{(12+2v)}{20} \rho_2 \left(\frac{a}{a_0}\right)^7 \right. \\
& \left. \left. + \frac{(53+5v)}{240} \rho_2 \left(\frac{a}{a_0}\right)^9 - 2(1+v) \rho_2^2 - 4v \rho_2 \left(\frac{a}{a_0}\right)^3 \right\} \right]
\end{aligned}$$

Non-uniform part of $\delta\bar{\sigma} - v\bar{\sigma}$ is

$$E\left(\frac{\rho}{R}\right)^2 \cos 2\theta \left\{ -\frac{(45-v)}{96} (f_1 + 6f_2)\left(\frac{\rho}{a}\right)^4 + \frac{(14-v)}{20} f_2\left(\frac{\rho}{a}\right)^6 - \frac{(45-3v)}{240} f_2\left(\frac{\rho}{a}\right)^8 + 2(1+v)p_2 + 12\lambda_2\left(\frac{\rho}{a}\right)^2 \right\}$$

$$\frac{1}{R} \frac{\partial \bar{\sigma}}{\partial \theta} = \left(\frac{\rho}{R}\right)^2 \cos 2\theta \left\{ \frac{1}{3} (f_1 + 2f_2)\left(\frac{\rho}{a}\right)^2 - \frac{(34+2v)}{96} (f_1 + 6f_2)\left(\frac{\rho}{a}\right)^4 + \frac{(31+v)}{20} f_2\left(\frac{\rho}{a}\right)^6 - \frac{(98+2v)}{240} f_2\left(\frac{\rho}{a}\right)^8 + 4(1+v)p_2 + 4(3+v)\lambda_2\left(\frac{\rho}{a}\right)^2 \right\}$$

$$\frac{\sigma}{R} = \left(\frac{\rho}{R}\right)^3 \sin \theta \left\{ \frac{1}{6} (f_1 + 2f_2)\left(\frac{\rho}{a}\right)^3 - \frac{(17+v)}{96} (f_1 + 6f_2)\left(\frac{\rho}{a}\right)^5 + \frac{(31+v)}{40} f_2\left(\frac{\rho}{a}\right)^7 - \frac{(49+v)}{240} f_2\left(\frac{\rho}{a}\right)^9 + 2(1+v)p_2\left(\frac{\rho}{a}\right) + 2(3+v)\lambda_2\left(\frac{\rho}{a}\right)^3 \right\}$$

$$\bar{M}_a = \cos 2\theta \left\{ -\frac{1}{96} f_1 - \frac{1}{40} f_2 - 2p_2 \right\}$$

$$\bar{\sigma}\bar{\sigma}_a = \cos 2\theta \left\{ -\frac{5}{32} f_1 - \frac{17}{40} f_2 + 2p_2 + 12\lambda_2 \right\}$$

$$\bar{r}\bar{\sigma}_a = \sin 2\theta \left\{ -\frac{7}{96} f_1 - \frac{7}{40} f_2 + 2p_2 + 6\lambda_2 \right\}$$

$$\left(\frac{u}{R}\right)_a = \cos 2\theta \left\{ -\frac{(13-3v)}{96} f_1 - \frac{26-26v}{140} f_2 - 2(1+v)p_2 - 12v\lambda_2 \right\}$$

$$\left(\frac{v}{R}\right)_a = \sin 2\theta \left\{ -\frac{(14v)}{96} f_1 - \frac{38+10v}{140} f_2 + 2(1+v)p_2 + 2(3+v)\lambda_2 \right\}$$

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$$\frac{1}{2} - 6r_2 - 4s_2 = \eta \left\{ -\frac{1}{96} f_1 - \frac{1}{40} f_2 - 2b_2 \right\}$$

$$6r_2 - \frac{1}{2} = \eta \left\{ -\frac{5}{32} f_1 - \frac{17}{40} f_2 + 2b_2 + 12n_2 \right\}$$

$$\frac{1}{2} + 6r_2 + 2s_2 = \eta \left\{ +\frac{5}{96} f_1 + \frac{7}{40} f_2 - 2b_2 - 6n_2 \right\}$$

$$\frac{1}{2}(1+\nu) + 2(1+\nu)r_2 + 4s_2 = \eta \left\{ -\frac{(13-3\nu)}{96} f_1 - \frac{26(1-\nu)}{240} f_2 - 2(1+\nu)b_2 - 4\nu n_2 \right\}$$

$$2(1+\nu)r_2 - \frac{1}{2}(1+\nu) = \eta \left\{ -\frac{(1+\nu)}{96} f_1 - \frac{38+10\nu}{240} f_2 + 2(1+\nu)b_2 + -(3+\nu)n_2 \right\}$$

$$\begin{array}{r} 12 \\ 2496 \\ \hline 41 \\ \hline 6240 \end{array}$$

$$r_2 + 0.6666667s_2 - 0.08333333 = \eta \left\{ +0.3333333p_2 + 0 + 0.00173611f_1 + 0.00416667f_2 \right\}$$

$$r_2 \quad 0 \quad -0.08333333 = \eta \left\{ +0.3333333p_2 + 2n_2 - 0.02604167f_1 - 0.02083333f_2 \right\}$$

$$r_2 + 0.3333333s_2 + 0.08333333 = \eta \left\{ -0.3333333p_2 - n_2 + 0.00868056f_1 + 0.02916667f_2 \right\}$$

$$r_2 + 1.53846154s_2 + 0.2500000 = \eta \left\{ -p_2 - 0.46153846n_2 - 0.04847756f_1 - 0.02916667f_2 \right\}$$

$$r_2 + 0 \quad -0.2500000 = \eta \left\{ +p_2 + 2.53846154n_2 - 0.00520833f_1 - 0.06570513f_2 \right\}$$

$$0.6666667s_2 + 0 = \eta \left\{ 0 \quad -2n_2 + 0.02777778f_1 + 0.07500000f_2 \right\}$$

$$0.3333333s_2 + 0.1666667 = \eta \left\{ -0.6666667p_2 - 3n_2 + 0.03472222f_1 + 0.10000000f_2 \right\}$$

$$1.2051282s_2 + 0.1666667 = \eta \left\{ -0.6666667p_2 + 0.53846154n_2 - 0.05715012f_1 - 0.05833333f_2 \right\}$$

$$1.53846154s_2 + 0.5000000 = \eta \left\{ -2p_2 - 3n_2 - 0.04326923f_1 + 0.03653846f_2 \right\}$$

$$\begin{aligned}
 f_2 + 0 &= \eta \left\{ 0 - 3f_2 + 0.04166667f_1 + 0.11250000f_2 \right\} \\
 f_2 + 0.50000 &= \eta \left\{ -2f_2 - 9f_2 + 0.104166667f_1 + 0.30000000f_2 \right\} \\
 f_2 + 0.13829787 &= \eta \left\{ -0.55319149f_2 + 0.44680851f_2 - 0.04742908f_1 - 0.04840425f_2 \right\} \\
 f_2 + 0.3250000 &= \eta \left\{ -1.3f_2 - 1.95f_2 - 0.02812500f_1 + 0.02375000f_2 \right\}
 \end{aligned}$$

$$\begin{aligned}
 0.5000000 &= \eta \left\{ -2f_2 - 6f_2 + 0.06250000f_1 + 0.18750000f_2 \right\} \\
 0.36170213 &= \eta \left\{ -1.44680851f_2 - 9.44680851f_2 + 0.15159575f_1 + 0.34640425f_2 \right\} \\
 0.18670213 &= \eta \left\{ -0.74680851f_2 - 2.39680851f_2 + 0.01930408f_1 + 0.07215425f_2 \right\}
 \end{aligned}$$

$$\begin{aligned}
 0.2500000 &= \eta \left\{ -f_2 - 3f_2 + 0.03125000f_1 + 0.09375000f_2 \right\} \\
 0.2500000 &= \eta \left\{ -f_2 - 6.52941176f_2 + 0.10477942f_1 + 0.24080852f_2 \right\} \\
 0.2500000 &= \eta \left\{ -f_2 - 3.20940171f_2 + 0.02584877f_1 + 0.09661680f_2 \right\}
 \end{aligned}$$

$$\begin{aligned}
 3.52941176 f_2 &= 0.07352942 f_1 + 0.14705882 f_2 \\
 3.32001005 f_2 &= 0.07893065 f_1 + 0.14419202 f_2
 \end{aligned}$$

$$\begin{aligned}
 3f_2 &= 0.06250000f_1 + 0.12500000f_2 \\
 3f_2 &= 0.07132266f_1 + 0.13029360f_2
 \end{aligned}$$

$$0 = 0.00882266f_1 + 0.00529360f_2$$

$$f_2 = -\frac{5}{3}f_1$$

failure !!!

a)

$$\left(\frac{w}{R}\right)_0 = \frac{1}{2} \left(\frac{R}{a}\right)^2 \left\{ 1 - \left(\frac{R}{a}\right)^2 \sin^2 \theta \right\}$$

$$\left(\frac{w}{R}\right) = \frac{1}{2} \left(\frac{R}{a}\right)^2 \left\{ 1 - \left(\frac{R}{a}\right)^2 \sin^2 \theta - \frac{1}{4} \left[1 - \left(\frac{R}{a}\right)^2 \right]^2 \right\}$$

$$\frac{1}{R} \frac{\partial w}{\partial R} = \frac{1}{2} \frac{1}{R} \left\{ -2 \sin^2 \theta + 12 \frac{1}{4} \left[1 - \left(\frac{R}{a}\right)^2 \right]^5 \right\}$$

$$\frac{1}{R} \frac{\partial w_0}{\partial R} = \frac{1}{2} \frac{1}{R} \left\{ -2 \sin^2 \theta \right\}$$

$$\frac{\partial^2 w}{\partial R^2} = \frac{1}{2} \frac{1}{R} \left\{ -2 \sin^2 \theta + 12 \frac{1}{4} \left[1 - \left(\frac{R}{a}\right)^2 \right]^5 - 120 \frac{1}{4} \left[1 - \left(\frac{R}{a}\right)^2 \right]^4 \left(\frac{R}{a}\right)^2 \right\}$$

$$= \frac{1}{2} \frac{1}{R} \left\{ -2 \sin^2 \theta + 12 \frac{1}{4} \left[1 - \left(\frac{R}{a}\right)^2 \right]^4 \left[1 - \left(\frac{R}{a}\right)^2 - 10 \left(\frac{R}{a}\right)^2 \right] \right\}$$

$$= \frac{1}{2} \frac{1}{R} \left\{ -2 \sin^2 \theta + 12 \frac{1}{4} \left[1 - \left(\frac{R}{a}\right)^2 \right]^4 \left[1 - 11 \left(\frac{R}{a}\right)^2 \right] \right\}$$

$$\frac{\partial^2 w_0}{\partial R^2} = \frac{1}{2} \frac{1}{R} \left\{ -2 \sin^2 \theta \right\}$$

$$\frac{1}{R^2} \frac{\partial w}{\partial \theta^2} = \frac{1}{R^2} \frac{\partial^2 w_0}{\partial \theta^2} = \frac{1}{R} (-\cos 2\theta)$$

$$- \left\{ \frac{1}{R^2} \frac{\partial \omega}{\partial R} \frac{\partial^2 \omega}{\partial R^2} - \frac{1}{R^2} \frac{\partial \omega_0}{\partial R} \frac{\partial^2 \omega_0}{\partial R^2} \right\}$$

6)

$$= \frac{1}{R^2} (\sin^2 \theta)^2 - \frac{1}{R^2} \left\{ \sin^2 \theta - 6 f_4 \left[1 - \left(\frac{R}{a} \right)^2 \right]^5 \right\} \left\{ \sin^2 \theta - 6 f_4 \left[1 - \left(\frac{R}{a} \right)^2 \right]^4 \left[1 - 11 \left(\frac{R}{a} \right)^2 \right] \right\}$$

$$= \frac{1}{R^2} \left\{ \underline{(1 - \cos^2 \theta) \cdot 6 f_4 \left[1 - \left(\frac{R}{a} \right)^2 \right]^4 \left[1 - 6 \left(\frac{R}{a} \right)^2 \right] - 36 f_4^2 \left[1 - \left(\frac{R}{a} \right)^2 \right]^7 \left[1 - 11 \left(\frac{R}{a} \right)^2 \right]} \right\}$$

$$- \left\{ \frac{1}{R^2} \frac{\partial^2 \omega}{\partial R^2} \frac{\partial^2 \omega}{\partial \theta^2} - \frac{1}{R^2} \frac{\partial^2 \omega_0}{\partial R^2} \frac{\partial^2 \omega_0}{\partial \theta^2} \right\}$$

$$= \frac{1}{R^2} \cos^2 \theta \left\{ \underline{6 f_4 \left[1 - \left(\frac{R}{a} \right)^2 \right]^4 \left[1 - 11 \left(\frac{R}{a} \right)^2 \right]} \right\}$$

The non-uniform terms in particular integral must be derived from

$$\cos^2 \theta \left[-6 f_4 \left[1 - \left(\frac{R}{a} \right)^2 \right]^4 5 \left(\frac{R}{a} \right)^2 \right] = - \left[30 f_4 \left[1 - \left(\frac{R}{a} \right)^2 \right]^4 \left(\frac{R}{a} \right)^2 \right] \cos^2 \theta$$

$$= -30 f_4 \cos^2 \theta \left\{ \left(\frac{R}{a} \right)^2 - 4 \left(\frac{R}{a} \right)^4 + 6 \left(\frac{R}{a} \right)^6 - 4 \left(\frac{R}{a} \right)^8 + \left(\frac{R}{a} \right)^{10} \right\}$$

$$\frac{\Phi}{R^2} = -30 f_4 \cos^2 \theta \left\{ \frac{1}{32 \cdot 12} \left(\frac{R}{a} \right)^6 - \frac{4}{60 \cdot 32} \left(\frac{R}{a} \right)^8 + \frac{6}{96 \cdot 60} \left(\frac{R}{a} \right)^{10} - \frac{4}{140 \cdot 96} \left(\frac{R}{a} \right)^{12} + \frac{1}{192 \cdot 140} \left(\frac{R}{a} \right)^{14} \right\} E \left(\frac{R}{R} \right)^2$$

$$\frac{1}{r} \frac{\partial \Phi}{\partial r} = -30 f_4 \cos 2\theta \left\{ \frac{1}{64} \left(\frac{r}{a}\right)^4 - \frac{1}{60} \left(\frac{r}{a}\right)^6 + \frac{1}{96} \left(\frac{r}{a}\right)^8 - \frac{1}{280} \left(\frac{r}{a}\right)^{10} \right. \quad c) \\ \left. + \frac{1}{1920} \left(\frac{r}{a}\right)^{12} \right\} E \left(\frac{a}{r}\right)^2$$

$$\frac{1}{r^2} \frac{\partial^2 \Phi}{\partial r^2} = +30 f_4 \cos 2\theta \left\{ \frac{1}{96} \left(\frac{r}{a}\right)^4 - \frac{1}{120} \left(\frac{r}{a}\right)^6 + \frac{1}{240} \left(\frac{r}{a}\right)^8 - \frac{1}{840} \left(\frac{r}{a}\right)^{10} \right. \\ \left. + \frac{1}{48 \times 140} \left(\frac{r}{a}\right)^{12} \right\} E \left(\frac{a}{r}\right)^2$$

$$\hat{r} \Phi = -30 f_4 \cos 2\theta \left\{ \frac{1}{192} \left(\frac{r}{a}\right)^4 - \frac{1}{120} \left(\frac{r}{a}\right)^6 + \frac{1}{160} \left(\frac{r}{a}\right)^8 - \frac{1}{420} \left(\frac{r}{a}\right)^{10} \right. \\ \left. + \frac{1}{192 \cdot 14} \left(\frac{r}{a}\right)^{12} \right\} E \left(\frac{a}{r}\right)^2$$

$$\hat{\theta} \Phi = -30 f_4 \cos 2\theta \left\{ \frac{5}{64} \left(\frac{r}{a}\right)^4 - \frac{7}{60} \left(\frac{r}{a}\right)^6 + \frac{9}{96} \left(\frac{r}{a}\right)^8 - \frac{11}{280} \left(\frac{r}{a}\right)^{10} + \frac{13}{1920} \left(\frac{r}{a}\right)^{12} \right\} E \left(\frac{a}{r}\right)^2$$

$$\hat{\theta} \Phi = -30 f_4 \sin 2\theta \left\{ \frac{5}{32 \cdot 6} \left(\frac{r}{a}\right)^4 - \frac{7}{60 \cdot 8} \left(\frac{r}{a}\right)^6 + \frac{9}{96 \cdot 5} \left(\frac{r}{a}\right)^8 - \frac{11}{140 \cdot 12} \left(\frac{r}{a}\right)^{10} \right. \\ \left. + \frac{13}{192 \cdot 70} \left(\frac{r}{a}\right)^{12} \right\} E \left(\frac{a}{r}\right)^2$$

$$\hat{r} \Phi - r \hat{\theta} \Phi = -30 f_4 \cos 2\theta \left\{ \frac{(1-15V)}{192} \left(\frac{r}{a}\right)^4 - \frac{(1-14V)}{120} \left(\frac{r}{a}\right)^6 + \frac{(1-15V)}{160} \left(\frac{r}{a}\right)^8 - \frac{(2-33V)}{840} \left(\frac{r}{a}\right)^{10} \right. \\ \left. + \frac{(5-91V)}{13440} \left(\frac{r}{a}\right)^{12} \right\} E \left(\frac{a}{r}\right)^2$$

$$\frac{1}{2} \left(\frac{\partial \Phi}{\partial r}\right)^2 - \frac{1}{2} \left(\frac{\partial \Phi}{\partial \theta}\right)^2 = \frac{1}{2} \left(\frac{a}{r}\right)^2 \left\{ -(1-\cos 2\theta) 6 f_4 \left[1 - \left(\frac{a}{r}\right)^2\right]^5 + 36 f_4^2 \left[1 - \left(\frac{a}{r}\right)^2\right]^{10} \right\} \\ = \frac{1}{2} \left(\frac{a}{r}\right)^2 \left\{ \cos 2\theta \cdot 6 f_4 \left[1 - \left(\frac{a}{r}\right)^2\right]^5 \left(\frac{a}{r}\right)^2 + \dots \right\}$$

The non-uniform part of $\frac{1}{2} \left(\frac{2u}{a} \right)^2 - \frac{1}{2} \left(\frac{2v}{a} \right)^2$

a)

$$3f_4 \cos 2\theta \left\{ \left(\frac{a}{a} \right)^2 - 5 \left(\frac{a}{a} \right)^4 + 10 \left(\frac{a}{a} \right)^6 - 10 \left(\frac{a}{a} \right)^8 + 5 \left(\frac{a}{a} \right)^{10} - \left(\frac{a}{a} \right)^{12} \right\} \left(\frac{a}{a} \right)^2$$

$$\frac{\partial u}{\partial r} = -30f_4 \cos 2\theta \left\{ \frac{1}{10} \left(\frac{a}{a} \right)^2 - \frac{(95+15v)}{192} \left(\frac{a}{a} \right)^4 + \frac{(119+14v)}{120} \left(\frac{a}{a} \right)^6 - \frac{(159+15v)}{160} \left(\frac{a}{a} \right)^8 \right. \\ \left. + \frac{(418+33v)}{840} \left(\frac{a}{a} \right)^{10} - \frac{(1339+91v)}{13440} \left(\frac{a}{a} \right)^{12} \right\} \left(\frac{a}{a} \right)^2$$

$$\frac{u}{r} = -30f_4 \cos 2\theta \left\{ \frac{1}{30} \left(\frac{a}{a} \right)^3 - \frac{(19+3v)}{192} \left(\frac{a}{a} \right)^5 + \frac{(17+2v)}{120} \left(\frac{a}{a} \right)^7 - \frac{(53+5v)}{480} \left(\frac{a}{a} \right)^9 \right. \\ \left. + \frac{(38+3v)}{840} \left(\frac{a}{a} \right)^{11} - \frac{(103+7v)}{13440} \left(\frac{a}{a} \right)^{13} \right\} \left(\frac{a}{a} \right)^3$$

$$\frac{v}{r} = -30f_4 \cos 2\theta \left\{ \frac{1}{30} \left(\frac{a}{a} \right)^3 - \frac{(19+3v)}{192} \left(\frac{a}{a} \right)^5 + \frac{(17+2v)}{120} \left(\frac{a}{a} \right)^7 - \frac{(53+5v)}{480} \left(\frac{a}{a} \right)^9 \right. \\ \left. + \frac{(38+3v)}{840} \left(\frac{a}{a} \right)^{11} - \frac{(103+7v)}{13440} \left(\frac{a}{a} \right)^{13} \right\} \left(\frac{a}{a} \right)^3$$

$$\frac{1}{E} (\bar{\sigma} - \bar{\sigma}_2) = -30f_4 \cos 2\theta \left\{ \frac{(15-v)}{192} \left(\frac{a}{a} \right)^4 - \frac{(14-v)}{120} \left(\frac{a}{a} \right)^6 + \frac{(15-v)}{160} \left(\frac{a}{a} \right)^8 - \frac{(33-2v)}{840} \left(\frac{a}{a} \right)^{10} \right. \\ \left. + \frac{(91-5v)}{13440} \left(\frac{a}{a} \right)^{12} \right\} \left(\frac{a}{a} \right)^2$$

$$\frac{\partial v}{\partial \theta} = -30f_4 \cos 2\theta \left\{ -\frac{1}{30} \left(\frac{a}{a} \right)^2 + \frac{(34+2v)}{192} \left(\frac{a}{a} \right)^4 - \frac{(31+v)}{120} \left(\frac{a}{a} \right)^6 + \frac{(49+v)}{240} \left(\frac{a}{a} \right)^8 \right. \\ \left. - \frac{(71+v)}{840} \left(\frac{a}{a} \right)^{10} + \frac{(97+v)}{6720} \left(\frac{a}{a} \right)^{12} \right\} \left(\frac{a}{a} \right)^2$$

$$\frac{y}{R} = -30f_4 \sin 2\theta \left\{ -\frac{1}{60} \left(\frac{a}{R}\right)^3 + \frac{(17+V)}{192} \left(\frac{a}{R}\right)^5 - \frac{(31+V)}{240} \left(\frac{a}{R}\right)^7 + \frac{(149+V)}{480} \left(\frac{a}{R}\right)^9 \right. \\ \left. - \frac{(7+V)}{1680} \left(\frac{a}{R}\right)^{11} + \frac{(97+V)}{13440} \left(\frac{a}{R}\right)^{13} \right\} \left(\frac{a}{R}\right)^2$$

$$\frac{1}{2} - 6f_2 - 4s_2 = \eta \left\{ -2p_2 - 0.03348214 f_2 \right\}$$

$$6f_2 - \frac{1}{2} = \eta \left\{ +2p_2 + 12s_2 - 0.27455357 f_2 \right\}$$

$$-\frac{1}{2} - 6f_2 - 2s_2 = \eta \left\{ +2p_2 + 6s_2 - 0.68080357 f_2 \right\}$$

$$\frac{1}{2}(1+V) + 2(1+V)f_2 + 4s_2 = \eta \left\{ -2(1+V)p_2 - 4V s_2 - 0.04479764 f_2 \right\}$$

$$2(1+V)f_2 - \frac{1}{2}(1+V) = \eta \left\{ +2(1+V)p_2 + 2(3+V)s_2 - 0.31584822 f_2 \right\}$$

$$f_2 + 0.6666667 s_2 - 0.0833333 = \eta \left\{ +0.333333 p_2 + 0 + 0.005580356 f_2 \right\}$$

$$f_2 - 0.0833333 = \eta \left\{ +0.333333 p_2 + 2s_2 - 0.045758928 f_2 \right\}$$

$$f_2 + 0.333333 s_2 + 0.0833333 = \eta \left\{ -0.333333 p_2 - s_2 + 0.113467262 f_2 \right\}$$

$$f_2 + 1.53846154 s_2 + 0.250000 = \eta \left\{ -p_2 - 0.461538462 s_2 - 0.01721701 f_2 \right\}$$

$$f_2 - 0.250000 = \eta \left\{ +p_2 + 2.53846154 s_2 - 0.12140001 f_2 \right\}$$

$$0.6666667 S_2 = \eta \left\{ 11.0671 - 2A_2 + 0.05133929 f_4 \right\} \quad f)$$

$$0.3333333 S_2 + 0.1666667 = \eta \left\{ -0.6666667 \beta_2 - 3A_2 + 0.15922619 f_4 \right\}$$

$$1.20512621 S_2 + 0.1666667 = \eta \left\{ -0.6666667 \beta_2 + 0.53846154 A_2 - 0.13069597 f_4 \right\}$$

$$1.53846154 S_2 + 0.500000 = \eta \left\{ -2\beta_2 - 3A_2 + 0.10425137 f_4 \right\}$$

$$S_2 + 0 = \eta \left\{ 0 - 3A_2 + 0.077001935 f_4 \right\}$$

$$S_2 + 0.5000 = \eta \left\{ -2\beta_2 - 9A_2 + 0.47767857 f_4 \right\}$$

$$S_2 + 0.1384787 = \eta \left\{ -0.55319149 \beta_2 + 0.44680851 A_2 - 0.10844985 f_4 \right\}$$

$$S_2 + 0.325000 = \eta \left\{ -1.30 \beta_2 - 1.95 A_2 + 0.06776339 f_4 \right\}$$

$$0.500000000 = \eta \left\{ -2\beta_2 - 6A_2 + 0.40066963 f_4 \right\}$$

$$0.36170213 = \eta \left\{ -1.44680851 \beta_2 - 9.44680851 A_2 + 0.58612842 f_4 \right\}$$

$$0.18670213 = \eta \left\{ -0.74680851 \beta_2 - 2.39680851 A_2 + 0.17621324 f_4 \right\}$$

$$0.2500000 = \eta \left\{ -\beta_2 - 3A_2 + 0.10033482 f_4 \right\}$$

$$0.250000 = \eta \left\{ -\beta_2 - 6.52941175 A_2 + 0.40511818 f_4 \right\}$$

$$0.250000 = \eta \left\{ -\beta_2 - 3.20940171 A_2 + 0.23595505 f_4 \right\}$$

$$3.52941175 \lambda_2 = 0.20478336 f_4$$

$$0.174065856 f_4 g)$$

$$3.31001005 \lambda_2 = 0.16916313 f_4$$

$$\frac{u}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left[1 - \left(\frac{x}{a} \right)^2 - \frac{f}{4} \left(1 + \cos \frac{\pi x}{a} \right) \left(1 + \cos \frac{\pi \lambda y}{a} \right) \right]$$

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$$\frac{u_0}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left[1 - \left(\frac{x}{a} \right)^2 \right]$$

$$R \frac{\partial^2 u}{\partial x^2} = \left[-1 + \frac{f}{8} \pi^2 \cos \frac{\pi x}{a} \left(1 + \cos \frac{\pi \lambda y}{a} \right) \right] \quad R \frac{\partial^2 u_0}{\partial x^2} = -1$$

$$R \frac{\partial^2 u}{\partial y^2} = \left[+ \frac{f}{8} \lambda^2 \pi^2 \left(1 + \cos \frac{\pi x}{a} \right) \cos \frac{\pi \lambda y}{a} \right]$$

$$R \frac{\partial^2 u}{\partial x \partial y} = \left[- \frac{f}{8} \lambda \pi^2 \sin \frac{\pi x}{a} \sin \frac{\pi \lambda y}{a} \right]$$

$$\nabla^4 F = \pi^2 \frac{E}{R^2} \left(\frac{f \lambda^2}{8} \right) \left[\frac{f \pi^2}{8} \frac{1}{4} \left(1 - \cos \frac{2\pi x}{a} \right) \left(1 - \cos \frac{2\pi \lambda y}{a} \right) \right.$$

$$\left. + \left(1 + \cos \frac{\pi x}{a} \right) \left(\cos \frac{\pi \lambda y}{a} \right) - \frac{f \pi^2}{8} \frac{1}{4} \left(1 + 2 \cos \frac{\pi x}{a} + \cos \frac{2\pi x}{a} \right) \left(1 + 2 \cos \frac{\pi \lambda y}{a} + \cos \frac{2\pi \lambda y}{a} \right) \right]$$

$$= \frac{E}{R^2} \left(\frac{f \lambda^2}{8} \right) \left[\cos \frac{\pi \lambda y}{a} + \cos \frac{\pi x}{a} \cos \frac{\pi \lambda y}{a} \right.$$

$$\left. + \frac{f \pi^2}{32} \left(1 - 2 \cos \frac{\pi x}{a} - 2 \cos \frac{2\pi x}{a} + \cos \frac{2\pi x}{a} \cos \frac{2\pi \lambda y}{a} \right) \right]$$

$$- 1 - 2 \cos \frac{\pi x}{a} - \cos \frac{2\pi x}{a} - 2 \cos \frac{\pi \lambda y}{a} - 2 \cos \frac{\pi x}{a} \cos \frac{\pi \lambda y}{a} - 2 \cos \frac{2\pi x}{a} \cos \frac{\pi \lambda y}{a}$$

$$- \cos \frac{2\pi \lambda y}{a} - 2 \cos \frac{\pi x}{a} \cos \frac{2\pi \lambda y}{a} - \cos \frac{2\pi x}{a} \cos \frac{2\pi \lambda y}{a} \left. \right]$$

$$= \frac{E}{R^2} \left(\frac{f \lambda^2}{8} \right) \left[- \frac{f \pi^2}{16} \cos \frac{\pi x}{a} + \left(1 - \frac{f \pi^2}{16} \right) \cos \frac{\pi \lambda y}{a} + \left(1 - \frac{f \pi^2}{8} \right) \cos \frac{\pi x}{a} \cos \frac{\pi \lambda y}{a} \right.$$

$$\left. - \frac{f \pi^2}{16} \left[\cos \frac{2\pi x}{a} + \cos^2 \frac{\pi \lambda y}{a} + \cos \frac{2\pi x}{a} \cos \frac{\pi \lambda y}{a} + \cos \frac{\pi x}{a} \cos \frac{2\pi \lambda y}{a} \right] \right]$$

$$\begin{aligned}
 F = E \left(\frac{a^2}{R} \right) \left(\frac{1+k^2}{8} \right) & \left[-\frac{1-k^2}{16} \frac{\cos \frac{\pi x}{a}}{\left(\frac{\pi}{a} \right)^2} + \frac{1}{16} \left(1 - \frac{1-k^2}{16} \right) \frac{\cos \frac{\pi x}{a}}{\left(\frac{\pi}{a} \right)^2} + \frac{1}{(1+k^2)^2} \left(1 - \frac{1-k^2}{8} \right) \frac{\cos \frac{\pi x}{a} \cos \frac{\pi y}{a}}{\left(\frac{\pi}{a} \right)^2} \right. \\
 & - \frac{1-k^2}{16} \left\{ \frac{1}{16} \frac{\cos \frac{2\pi x}{a}}{\left(\frac{\pi}{a} \right)^2} + \frac{1}{16k^2} \frac{\cos \frac{2\pi y}{a}}{\left(\frac{\pi}{a} \right)^2} + \frac{1}{(1+4k^2)^2} \frac{\cos \frac{\pi x}{a} \cos \frac{2\pi y}{a}}{\left(\frac{\pi}{a} \right)^2} + \frac{1}{(4+k^2)^2} \frac{\cos \frac{2\pi x}{a} \cos \frac{\pi y}{a}}{\left(\frac{\pi}{a} \right)^2} \right\} \\
 & + \frac{1}{\left(\frac{\pi}{a} \right)^2} \left\{ a_1 \cosh \frac{\pi y}{a} + b_1 \left(\frac{\pi y}{a} \right) \sinh \frac{\pi y}{a} \right\} \cos \frac{\pi x}{a} \\
 & + \frac{1}{\left(\frac{\pi}{a} \right)^2} \left\{ a_2 \cosh \left(\frac{2\pi y}{a} \right) + b_2 \left(\frac{2\pi y}{a} \right) \sinh \frac{2\pi y}{a} \right\} \cos \frac{2\pi x}{a} \left. \right] + \frac{Q}{2} x^2
 \end{aligned}$$

$$\begin{aligned}
 \frac{Q_1}{E} = \left(\frac{a^2}{R} \right) \left(\frac{1+k^2}{8} \right) & \left[-\left\{ a_1 \cosh \frac{\pi y}{a} + b_1 \left(\frac{\pi y}{a} \right) \sinh \frac{\pi y}{a} \right\} \cos \frac{\pi x}{a} - \left\{ a_2 \cosh \frac{2\pi y}{a} + b_2 \left(\frac{2\pi y}{a} \right) \sinh \frac{2\pi y}{a} \right\} \cos \frac{2\pi x}{a} \right. \\
 & + \frac{1-k^2}{16} \cos \frac{\pi x}{a} + \frac{1}{(1+k^2)^2} \left(\frac{1-k^2}{8} - 1 \right) \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} + \frac{1-k^2}{16} \left\{ \frac{1}{4} \frac{\cos \frac{2\pi x}{a}}{a} + \frac{1}{(1+4k^2)^2} \cos \frac{\pi x}{a} \cos \frac{2\pi y}{a} \right. \\
 & \left. \left. + \frac{4}{(4+k^2)^2} \cos \frac{2\pi x}{a} \cos \frac{\pi y}{a} \right\} \right] + \frac{Q}{E}
 \end{aligned}$$

$$a_1 \cosh \pi + \pi b_1 \sinh \pi = \frac{1-k^2}{16} + \frac{1}{(1+k^2)^2} \left(\frac{1-k^2}{8} - 1 \right) + \frac{1}{(1+k^2)^2} \frac{1-k^2}{16}$$

$$(a_1 + b_1) \sinh \pi + \pi b_1 \cosh \pi = 0$$

$$a_2 \cosh 2\pi + 2\pi b_2 \sinh 2\pi = \frac{1-k^2}{16} \left[\frac{1}{4} + \frac{4}{(4+k^2)^2} \right]$$

$$(a_2 + b_2) \sinh 2\pi + 2\pi b_2 \cosh 2\pi = 0$$

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$$\frac{q_2}{E} = \left(\frac{a^2}{R}\right) \left(\frac{h^2}{g}\right) \left[\left\{ \frac{h^2}{16} + \frac{1}{(1+h^2)^2} \left(\frac{h^2}{g} - 1 \right) \cos \frac{\pi h}{a} + \frac{h^2}{16} \cos \frac{2\pi h}{a} - a_1 \cos \frac{\pi h}{a} - b_1 \left(\frac{\pi h}{a} \right) \sinh \frac{\pi h}{a} \right\} \cos \frac{2\pi x}{a} \right. \\ \left. + \left\{ \frac{h^2}{16} + \frac{h^2}{16} \frac{4}{(4+h^2)^2} \cos \frac{\pi h}{a} - a_2 \cosh \frac{2\pi h}{a} - b_2 \left(\frac{2\pi h}{a} \right) \sinh \frac{2\pi h}{a} \right\} \cos \frac{2\pi x}{a} \right] - \frac{E}{E}$$

$$\frac{1}{4\pi} \int_0^\pi \sigma_y^2 dx = \frac{1}{2} \left(\frac{a}{R}\right)^4 \left(\frac{h^2}{g}\right)^2 \left[\left\{ \frac{h^2}{16} + \frac{1}{(1+h^2)^2} \left(\frac{h^2}{g} - 1 \right) \cos \frac{\pi h}{a} + \frac{h^2}{16} \cos \frac{2\pi h}{a} - \left\{ a_1 \cosh \frac{\pi h}{a} + b_1 \left(\frac{\pi h}{a} \right) \sinh \frac{\pi h}{a} \right\}^2 \right. \right. \\ \left. \left. + \frac{1}{2} \left(\frac{a}{R}\right)^4 \left(\frac{h^2}{g}\right)^2 \left[\left\{ \frac{h^2}{16} + \frac{h^2}{16} \frac{4}{(4+h^2)^2} \cos \frac{\pi h}{a} - \left\{ a_2 \cosh \frac{2\pi h}{a} + b_2 \left(\frac{2\pi h}{a} \right) \sinh \frac{2\pi h}{a} \right\}^2 \right\} + \left(\frac{E}{E} \right)^2 \right] \right]$$

$$\frac{1}{abE^2} \int_0^a \int_0^b \sigma_y^2 dx dy = \left(\frac{a}{R}\right)^4 \left(\frac{h^2}{g}\right)^2 \left[\frac{1}{2} \left(\frac{h^2}{16}\right)^2 + \frac{1}{4} \frac{1}{(1+h^2)^4} \left(\frac{h^2}{g} - 1 \right)^2 + \frac{1}{4} \frac{1}{(1+4h^2)^4} \left(\frac{h^2}{16} \right)^2 + \frac{1}{4} \frac{1}{(1+h^2)^4} \left(\frac{h^2}{g} \right)^2 \right] \\ - \left(\frac{a}{R}\right)^4 \left(\frac{h^2}{g}\right)^2 \left[\frac{h^2}{16} \frac{1}{\pi} \int_0^\pi (a_1 \cosh u + b_1 u \sinh u) du + \frac{1}{(1+h^2)^2} \left(\frac{h^2}{g} - 1 \right) \frac{1}{\pi} \int_0^\pi \cos u (a_1 \cosh u + b_1 u \sinh u) du \right. \\ \left. + \frac{1}{(1+4h^2)^2} \frac{h^2}{16} \frac{1}{\pi} \int_0^\pi \cos 2u (a_1 \cosh u + b_1 u \sinh u) du + \frac{h^2}{64} \frac{1}{\pi} \int_0^\pi (a_2 \cosh 2u + b_2 (2u) \sinh 2u) du \right. \\ \left. + \frac{h^2}{4(4+h^2)^2} \frac{1}{\pi} \int_0^\pi \cos u (a_2 \cosh 2u + b_2 (2u) \sinh 2u) du \right] \\ + \left(\frac{a}{R}\right)^4 \left(\frac{h^2}{g}\right)^2 \left[\frac{1}{2} \frac{1}{\pi} \int_0^\pi (a_1 \cosh u + b_1 u \sinh u)^2 du + \frac{1}{2\pi} \int_0^\pi (a_2 \cosh 2u + b_2 (2u) \sinh 2u)^2 du \right]$$

We have the following integrals:

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$$\frac{1}{\pi} \int_0^{\pi} (x \cosh u + \beta u \sinh u) du = (x - \beta) \frac{\sinh \pi}{\pi} + \beta \cosh \pi$$

$$\frac{1}{\pi} \int_0^{\pi} \cos u (x \cosh u + \beta u \sinh u) du = -x \frac{\sinh \pi}{\pi} - \beta \cosh \pi$$

$$\frac{1}{\pi} \int_0^{\pi} \cos 2u (x \cosh u + \beta u \sinh u) du = \frac{x}{5} \frac{\sinh \pi}{\pi} + \beta \left(\frac{1}{5} \cosh \pi + \frac{3}{25} \frac{\sinh \pi}{\pi} \right)$$

$$\frac{1}{\pi} \int_0^{\pi} (\gamma \cosh 2u + \delta 2u \sinh 2u) du = (\gamma - \delta) \frac{\sinh 2\pi}{2\pi} + \delta \cosh 2\pi$$

$$\frac{1}{\pi} \int_0^{\pi} \cos u (\gamma \cosh 2u + \delta 2u \sinh 2u) du = \frac{4}{5} \gamma \frac{\sinh 2\pi}{2\pi} + \delta \left(\frac{4}{5} \cosh 2\pi + \frac{12}{25} \frac{\sinh 2\pi}{2\pi} \right)$$

$$\frac{1}{\pi} \int_0^{\pi} \left\{ x \cosh u + \beta u \sinh u \right\}^2 du$$

$$= \frac{x^2}{2} \left(\frac{\sinh 2\pi}{2\pi} + 1 \right) + \frac{x\beta}{2} \left(\cosh 2\pi - \frac{\sinh 2\pi}{2\pi} \right) + \frac{\beta^2}{2} \left\{ \left(\frac{2\pi+1}{2} \right) \frac{\sinh 2\pi}{2\pi} - \frac{1}{2} \cosh 2\pi - \frac{\pi^2}{3} \right\}$$

$$\frac{1}{\pi} \int_0^{\pi} \left\{ \gamma \cosh 2u + \delta 2u \sinh 2u \right\}^2 du$$

$$= \frac{\gamma^2}{2} \left(\frac{\sinh 4\pi}{4\pi} + 1 \right) + \frac{\gamma\delta}{2} \left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi} \right) + \frac{\delta^2}{2} \left\{ \left(\frac{4\pi+1}{2} \right) \frac{\sinh 4\pi}{4\pi} - \frac{1}{2} \cosh 4\pi - \frac{4\pi^2}{3} \right\}$$

$$\frac{1}{\pi} \int_0^{\pi} \sin u \sinh u \, du = \frac{1}{2} \frac{\sinh \pi}{\pi}$$

$$\frac{1}{\pi} \int_0^{\pi} \sin u \cdot u \cosh u \, du = \frac{1}{2} \left(\cosh \pi - \frac{\sinh \pi}{\pi} \right)$$

$$\frac{1}{\pi} \int_0^{\pi} \sin 2u \cdot \sinh u \, du = -\frac{2}{5} \left(\frac{\sinh \pi}{\pi} \right)$$

$$\frac{1}{\pi} \int_0^{\pi} \sin u \sinh 2u \, du = \frac{2}{5} \left(\frac{\sinh 2\pi}{2\pi} \right)$$

$$\frac{1}{\pi} \int_0^{\pi} 2u \cdot \sin u \cosh 2u \, du = \frac{2}{5} \cosh 2\pi - \frac{16}{25} \frac{\sinh 2\pi}{2\pi}$$

$$\frac{1}{\pi} \int_0^{\pi} u \cdot \sin 2u \cosh u \, du = -\frac{2}{5} \cosh \pi + \frac{6}{25\pi} \frac{\sinh \pi}{\pi}$$

$$\frac{1}{\pi} \int_0^{\pi} (\alpha \sinh u + \beta u \cosh u)^2 \, du$$

$$= \frac{\alpha^2}{2} \left(\frac{\sinh 2\pi}{2\pi} - 1 \right) + \frac{\alpha\beta}{2} \left(\cosh 2\pi - \frac{\sinh 2\pi}{2\pi} \right) + \frac{\beta^2}{2} \left(\frac{2\pi^2+1}{2} \frac{\sinh 2\pi}{2\pi} - \frac{1}{2} \cosh 2\pi + \frac{\pi^2}{3} \right)$$

$$\frac{1}{\pi} \int_0^{\pi} (\gamma \sinh 2u + \delta u \cosh 2u)^2 \, du$$

$$= \frac{\gamma^2}{2} \left(\frac{\sinh 4\pi}{4\pi} - 1 \right) + \frac{\gamma\delta}{2} \left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi} \right) + \frac{\delta^2}{2} \left(\frac{8\pi^2+1}{2} \frac{\sinh 4\pi}{4\pi} - \frac{1}{2} \cosh 4\pi + \frac{4\pi^2}{3} \right)$$

Therefore

$$\begin{aligned}
 \frac{1}{abE^2} \int_0^a \int_0^b y^2 dx dy &= \left(\frac{a^2}{16} \right) \left(\frac{b^2}{8} \right) \left[\frac{1}{2} \left(\frac{b^2}{16} \right)^2 + \frac{1}{4(1+b^2)^2} \left(\frac{b^2}{8} - 1 \right)^2 + \frac{1}{4(1+b^2)^2} \left(\frac{b^2}{16} \right)^2 + \frac{1}{4(1+b^2)^2} \left(\frac{b^2}{8} \right)^2 \right] \\
 &- \left\{ \frac{b^2}{16} [(a_1 - b_1) \frac{\sin b_1 \pi}{\pi} + b_1 \cosh \pi] + \frac{1}{(1+b^2)^2} \left(\frac{b^2}{8} - 1 \right) \left[-a_1 \frac{\sin b_1 \pi}{\pi} - b_1 \cosh \pi \right] \right. \\
 &+ \frac{1}{(1+4b^2)^2} \left[a_1 \frac{1}{5} \frac{\sin b_1 \pi}{\pi} + b_1 \left(\frac{1}{5} \cosh \pi + \frac{3}{25} \frac{\sin b_1 \pi}{\pi} \right) \right] + \frac{b^2}{64} [(a_2 - b_2) \frac{\sin b_2 \pi}{2\pi} + b_2 \cosh 2\pi] \\
 &+ \frac{(4\pi^2)}{4(4+b^2)^2} \left[-\frac{4}{5} a_2 \frac{\sin b_2 \pi}{2\pi} + b_2 \left(-\frac{4}{5} \cosh 2\pi + \frac{12}{25} \frac{\sin b_2 \pi}{2\pi} \right) \right] \Bigg\} \\
 &+ \frac{1}{4} a_1^2 \left(\frac{\sin b_2 \pi}{2\pi} + 1 \right) + \frac{a_1 b_1}{4} \left(\cosh 2\pi - \frac{\sin b_2 \pi}{2\pi} \right) + \frac{b_1^2}{4} \left(\frac{2\pi^2 + 1}{2} \frac{\sin b_2 \pi}{2\pi} - \frac{1}{2} \cosh 2\pi - \frac{\pi^2}{3} \right) \\
 &+ \frac{1}{4} a_2^2 \left(\frac{\sin b_4 \pi}{4\pi} + 1 \right) + \frac{a_2 b_2}{4} \left(\cosh 4\pi - \frac{\sin b_4 \pi}{4\pi} \right) + \frac{b_2^2}{4} \left(\frac{8\pi^2 + 1}{2} \frac{\sin b_4 \pi}{4\pi} - \frac{1}{2} \cosh 4\pi - \frac{4\pi^2}{3} \right) \Bigg]
 \end{aligned}$$

$$\begin{aligned} \frac{\partial \psi}{\partial x} \Big|_{x=0} = \frac{1}{R} \left(\frac{1}{g} \right)^{1/2} \left[\left\{ (a_1 + 2b_1) \cosh \frac{\pi \lambda y}{a} + b_1 \left(\frac{\pi \lambda y}{a} \right) \sinh \frac{\pi \lambda y}{a} \right\} \cosh \frac{2\pi x}{a} + \left\{ (a_2 + 2b_2) \cosh \frac{2\pi x}{a} + b_2 \left(\frac{2\pi x}{a} \right) \sinh \frac{2\pi x}{a} \right\} \cosh \frac{2\pi y}{a} \right. \\ \left. + \frac{1}{\lambda^2} \left(\frac{\pi^2}{16} - 1 \right) \cosh \frac{\pi \lambda y}{a} + \frac{1}{(1+\lambda^2)^2} \left(\frac{\pi^2}{8} - 1 \right) \cosh \frac{2\pi x}{a} + \frac{\pi^2}{16} \left\{ \frac{1}{4\lambda^2} \cosh \frac{\pi \lambda y}{a} + \frac{\pi^2}{16} \cosh \frac{2\pi x}{a} + \frac{b_1}{(1+\lambda^2)^2} \cosh \frac{2\pi y}{a} \right. \right. \\ \left. \left. + \frac{1}{(4+\lambda^2)^2} \cosh \frac{2\pi x}{a} \cosh \frac{\pi \lambda y}{a} \right\} \right] \end{aligned}$$

$$\begin{aligned} = \lambda^2 \left(\frac{g}{R} \right)^{1/2} \left(\frac{\pi^2}{8} \right)^{1/2} \left[\left\{ \frac{1}{\lambda^2} \left(\frac{\pi^2}{16} - 1 \right) \cosh \frac{\pi \lambda y}{a} + \frac{\pi^2}{16\lambda^4} \cosh \frac{2\pi x}{a} \right\} \right. \\ \left. + \left\{ \frac{1}{(1+\lambda^2)^2} \left(\frac{\pi^2}{8} - 1 \right) \cosh \frac{\pi \lambda y}{a} + \frac{\pi^2}{4(1+\lambda^2)^2} \cosh \frac{2\pi x}{a} + (a_1 + 2b_1) \cosh \frac{\pi \lambda y}{a} + b_1 \left(\frac{\pi \lambda y}{a} \right) \sinh \frac{\pi \lambda y}{a} \right\} \cosh \frac{2\pi x}{a} \right. \\ \left. + \left\{ \frac{\pi^2}{16(4+\lambda^2)^2} \cosh \frac{\pi \lambda y}{a} + (a_2 + 2b_2) \cosh \frac{2\pi x}{a} + b_2 \left(\frac{2\pi x}{a} \right) \sinh \frac{2\pi x}{a} \right\} \cosh \frac{2\pi y}{a} \right] \end{aligned}$$

$$\begin{aligned}
\frac{1}{abE^2} \int_0^a \int_0^b \sigma_x^2 dx dy &= h^4 \left(\frac{a}{R} \right)^4 \left(\frac{b^2}{8} \right)^2 \left[\frac{1}{2} \frac{1}{b^8} \left(\frac{b^2}{16} - 1 \right)^2 + \frac{1}{2} \frac{1}{b^8} \left(\frac{b^2}{64} \right)^2 + \frac{1}{4(1+h^2)^6} \left(\frac{b^2}{8} - 1 \right)^2 + \frac{1}{4(1+h^2)^6} \left(\frac{b^2}{4} \right)^2 \right. \\
&\quad \left. + \frac{1}{4(4+h^2)^6} \left(\frac{b^2}{16} \right)^2 \right] \\
&+ \frac{1}{(1+h^2)^2} \left(\frac{b^2}{8} - 1 \right) \left\{ - (a_1 + 2b_1) \frac{\sin b_1 \pi}{\pi} - b_1 \cos \pi \right\} + \frac{1}{4(1+h^2)^2} \left\{ \frac{a_1 + 2b_1}{5} \frac{\sin b_1 \pi}{\pi} + b_1 \left(\frac{1}{5} \cos \pi + \frac{3}{25} \frac{\sin 2\pi}{\pi} \right) \right\} \\
&+ \frac{1}{16(4+h^2)^2} \left\{ - \frac{4}{5} (a_2 + 2b_2) \frac{\sin b_2 \pi}{2\pi} + b_2 \left(-\frac{4}{5} \cos 2\pi + \frac{12}{25} \frac{\sin 2\pi}{2\pi} \right) \right\} \\
&+ \frac{1}{4} (a_1 + 2b_1)^2 \left(\frac{\sin b_1 2\pi}{2\pi} + 1 \right) + \frac{(a_1 + 2b_1) b_1}{4} \left(\cos b_1 \pi - \frac{\sin b_1 2\pi}{2\pi} \right) + \frac{b_1^2}{4} \left(\frac{2\pi^2}{2} \frac{\sin b_1 2\pi}{2\pi} - \frac{1}{2} \cos 2\pi - \frac{\pi^2}{3} \right) \\
&+ \frac{1}{4} (a_2 + 2b_2)^2 \left(\frac{\sin b_2 4\pi}{4\pi} + 1 \right) + \frac{(a_2 + 2b_2) b_2}{4} \left(\cos b_2 \pi - \frac{\sin b_2 4\pi}{4\pi} \right) + \frac{b_2^2}{4} \left(\frac{8\pi^2}{2} \frac{\sin b_2 4\pi}{4\pi} - \frac{1}{2} \cos 4\pi - \frac{4\pi^2}{3} \right) \Bigg]
\end{aligned}$$

$$\begin{aligned}
\frac{v_y}{E} &= \lambda \left(\frac{b_1}{R} \right)^2 \left(\frac{1 + \nu}{8} \right) \left[\left\{ (a_1 + b_1) \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} + b_1 \left(\frac{1 + \nu}{a} \right) \cos \frac{\pi x}{a} \right\} \sin \frac{\pi y}{a} + \left\{ (a_2 + b_2) \sinh \frac{2\pi y}{a} + b_2 \left(\frac{2\pi y}{a} \right) \cosh \frac{2\pi y}{a} \right\} \sin \frac{\pi x}{a} \right. \\
&\quad \left. + \frac{1}{(1 + \nu)^2} \left(\frac{1 + \nu}{8} - 1 \right) \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} + \frac{1 + \nu}{8} \frac{1}{(1 + \nu)^2} \sin \frac{\pi x}{a} \sin \frac{2\pi y}{a} + \frac{1 + \nu}{8} \frac{1}{(1 + \nu)^2} \sin \frac{2\pi x}{a} \sin \frac{\pi y}{a} \right] \\
&= \lambda \left(\frac{b_1}{R} \right)^2 \left(\frac{1 + \nu}{8} \right) \left[\left\{ \frac{1}{(1 + \nu)^2} \left(\frac{1 + \nu}{8} - 1 \right) \sin \frac{\pi x}{a} + \frac{1 + \nu}{8} \sin \frac{2\pi x}{a} + (a_1 + b_1) \sinh \frac{\pi x}{a} + b_1 \left(\frac{1 + \nu}{a} \right) \cosh \frac{\pi x}{a} \right\} \sin \frac{\pi y}{a} \right. \\
&\quad \left. + \left\{ \frac{1}{(1 + \nu)^2} \left(\frac{1 + \nu}{8} \right) \sin \frac{\pi x}{a} + (a_2 + b_2) \sinh \frac{2\pi y}{a} + b_2 \left(\frac{2\pi y}{a} \right) \cosh \frac{2\pi y}{a} \right\} \sin \frac{2\pi x}{a} \right]
\end{aligned}$$

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$$\begin{aligned}
\frac{1}{abE^2} \int_0^a \int_0^b xy^2 dy dx &= \lambda^2 \left(\frac{a^4}{b} \right) \left(\frac{b^2}{8} \right) \left[\frac{1}{4(1+\lambda^2)^2} \left(\frac{b^2}{8} - 1 \right)^2 + \frac{1}{4(1+\lambda^2)^2} \left(\frac{b^2}{8} \right)^2 + \frac{1}{4(4+\lambda^2)^2} \left(\frac{b^2}{8} \right)^2 \right] \\
&+ \frac{1}{(1+\lambda^2)^2} \left(\frac{b^2}{8} - 1 \right) \left\{ \frac{1}{2} (a_1 + b_1) \frac{\sin \frac{b^2}{8}}{\pi} + \frac{b_1}{2} \left(\cos b^2 - \frac{\sin b^2}{\pi} \right) \right\} + \frac{1}{(1+\lambda^2)^2} \frac{b^2}{8} \left\{ -\frac{2}{5} (a_1 + b_1) \frac{\sin b^2}{\pi} + b_1 \left(-\frac{2}{5} \cos b^2 + \frac{4}{25\pi} \right) \right\} \\
&+ \frac{1}{(4+\lambda^2)^2} \left(\frac{b^2}{8} \right) \left\{ \frac{2}{5} (a_2 + b_2) \frac{\sin \frac{b^2}{8}}{2\pi} + b_2 \left(\frac{2}{5} \cos 2\pi - \frac{4}{25} \frac{\sin 2\pi}{2\pi} \right) \right\} \\
&+ \frac{(a_1 + b_1)^2}{4} \left(\frac{\sin b^2}{2\pi} - 1 \right) + \frac{(a_1 + b_1)b_1}{4} \left(\cos b^2 - \frac{\sin b^2}{2\pi} \right) + \frac{b_1^2}{4} \left(\frac{2\pi^2 + 1}{2} \frac{\sin b^2}{2\pi} - \frac{1}{2} \cos 2\pi + \frac{\pi^2}{3} \right) \\
&+ \frac{(a_2 + b_2)^2}{4} \left(\frac{\sin \frac{b^2}{4\pi}}{4\pi} - 1 \right) + \frac{(a_2 + b_2)b_2}{4} \left(\cos \frac{b^2}{4\pi} - \frac{\sin \frac{b^2}{4\pi}}{4\pi} \right) + \frac{b_2^2}{4} \left(\frac{8\pi^2 + 1}{2} \frac{\sin \frac{b^2}{4\pi}}{4\pi} - \frac{1}{2} \cos 4\pi + \frac{4\pi^2}{3} \right) \Bigg]
\end{aligned}$$

$$\begin{aligned}
 F_1 = (f\pi^2)^2 & \left\{ \frac{1}{512} \frac{1}{\lambda^4} + \frac{1}{8192} \frac{1}{\lambda^4} + \frac{1}{256} \frac{1}{(1+\lambda^2)^4} + \frac{1}{84(1+\lambda^2)^4} + \frac{\lambda^4}{1024(4+\lambda^2)^4} \right. \\
 & + \frac{1}{512} + \frac{1}{8192} + \frac{1}{256} \frac{1}{(1+\lambda^2)^2} + \frac{1}{1024(1+4\lambda^2)^4} + \frac{1}{64(4+\lambda^2)^2} + \frac{2\lambda^2}{256(1+\lambda^2)^4} + \left. \frac{2\lambda^2}{256(4+\lambda^2)^2} \right\} \\
 - (f\pi^2) & \left\{ \frac{1}{16} \frac{1}{\lambda^4} + \frac{\lambda^4}{16(1+\lambda^2)^4} + \frac{1}{16(1+\lambda^2)^4} + \frac{2\lambda^2}{16(1+\lambda^2)^2} \right\} + \left\{ \frac{1}{2\lambda^4} + \frac{\lambda^4}{4(1+\lambda^2)^4} + \frac{1}{4(1+\lambda^2)^4} + \frac{2\lambda^2}{4(1+\lambda^2)^4} \right\}
 \end{aligned}$$

$$= (f\pi^2)^2 \left\{ \frac{17}{8192} \left(1 + \frac{1}{\lambda^4}\right) + \frac{1}{256} \frac{1}{(1+\lambda^2)^2} + \frac{1}{1024} \frac{1}{(1+4\lambda^2)^2} + \frac{1}{1024(4+\lambda^2)^2} \right\}$$

$$- (f\pi^2) \left\{ \frac{1}{16\lambda^4} + \frac{1}{16(1+\lambda^2)^2} \right\} + \left\{ \frac{1}{2\lambda^4} + \frac{1}{4(1+\lambda^2)^2} \right\}$$

$$a, \frac{\sinh \pi}{\pi}$$

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$$\begin{aligned}
 & - \frac{\lambda^4}{(1+\lambda^2)^2} \left(\frac{f\pi^2}{8} - 1 \right) + \frac{\lambda^4}{20(1+4\lambda^2)^2} (f\pi^2) - \frac{f\pi^2}{16} + \frac{1}{(1+\lambda^2)^2} \left(\frac{f\pi^2}{8} - 1 \right) \\
 & - \frac{1}{80(1+4\lambda^2)^2} (f\pi^2) + \frac{\lambda^2}{(1+\lambda^2)^2} \left(\frac{f\pi^2}{8} - 1 \right) - \frac{\lambda^2 (f\pi^2)}{20(1+4\lambda^2)^2} = g_1(\lambda)
 \end{aligned}$$

$$b, \frac{\sinh \pi}{\pi}$$

$$\begin{aligned}
 & - \frac{2\lambda^4}{(1+\lambda^2)^2} \left(\frac{f\pi^2}{8} - 1 \right) + \frac{13}{25} \frac{\lambda^4}{4(1+4\lambda^2)^2} (f\pi^2) + \frac{f\pi^2}{16} - \frac{3}{25} \frac{1}{16(1+4\lambda^2)^2} (f\pi^2) \\
 & - \frac{3}{25} \frac{\lambda^2 f\pi^2}{2(1+4\lambda^2)^2} = g_2(\lambda)
 \end{aligned}$$

$$b, \cosh \pi$$

$$\begin{aligned}
 & - \frac{\lambda^4}{(1+\lambda^2)^2} \left(\frac{f\pi^2}{8} - 1 \right) + \frac{\lambda^4 f\pi^2}{20(1+4\lambda^2)^2} - \frac{f\pi^2}{16} + \frac{1}{(1+\lambda^2)^2} \left(\frac{f\pi^2}{8} - 1 \right) - \frac{f\pi^2}{80(1+4\lambda^2)^2} \\
 & + \frac{\lambda^2}{(1+\lambda^2)^2} - \frac{\lambda^2}{10(1+4\lambda^2)^2} (f\pi^2) = g_1(\lambda)
 \end{aligned}$$

$$a_2, \frac{\sinh 2\pi}{2\pi}$$

$$- \frac{\lambda^4 (f\pi^2)}{20(4+\lambda^2)^2} - \frac{f\pi^2}{64} + \frac{f\pi^2}{5(4+\lambda^2)^2} + \frac{\lambda^2 (f\pi^2)}{10(4+\lambda^2)^2} = g_3(\lambda)$$

$$\underline{\underline{b_2 \frac{\sinh 2\pi}{2\pi}}}$$

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$$- \frac{\frac{1}{8} \lambda^2}{25 \times 4 (4+\lambda^2)^2} (f\pi^2) + \frac{f\pi^2}{64} - \frac{12}{100} \frac{(f\pi^2)}{(4+\lambda^2)^2} - \frac{3 \times 2 \lambda^2}{25 \times 4 (4+\lambda^2)^2} (f\pi^2) = g_4'(\lambda)$$

$$b_2 \cosh 2\pi$$

$$- \frac{\lambda^4 (f\pi^2)}{20 (4+\lambda^2)^2} - \frac{f\pi^2}{64} + \frac{(f\pi^2)}{5 (4+\lambda^2)^2} + \frac{4 \lambda^2 (f\pi^2)}{40 (4+\lambda^2)^2} = g_3(\lambda)$$

$$\cosh \pi a_1 + \pi \sinh \pi b_1 = \left\{ \frac{1}{16} + \frac{1}{8(1+\lambda^2)^2} + \frac{1}{16(1+4\lambda^2)^2} \right\} (f\pi^2) - \frac{1}{(1+\lambda^2)^2}$$

$$\sinh \pi a_1 + (\sinh \pi + \pi \cosh \pi) b_1 = 0$$

$$b_1 = - \frac{\frac{\sinh \pi}{\pi}}{\frac{\sinh 2\pi}{2\pi} + 1} \left[\left\{ \frac{1}{16} + \frac{1}{8(1+\lambda^2)^2} + \frac{1}{16(1+4\lambda^2)^2} \right\} f\pi^2 - \frac{1}{(1+\lambda^2)^2} \right] \frac{H_1(\lambda)}{H_2(\lambda)}$$

$$a_1 = + \frac{\frac{\sinh \pi}{\pi} + \cosh \pi}{\frac{\sinh 2\pi}{2\pi} + 1} \left[\left\{ \frac{1}{16} + \frac{1}{8(1+\lambda^2)^2} + \frac{1}{16(1+4\lambda^2)^2} \right\} f\pi^2 - \frac{1}{(1+\lambda^2)^2} \right]$$

$$a_2 = + \frac{\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi}{\frac{\sinh 4\pi}{4\pi} + 1} \left[\frac{1}{64} + \frac{1}{4(4+\lambda^2)^2} \right] f\pi^2 \frac{H_3(\lambda)}{H_4(\lambda)}$$

$$b_2 = - \frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi} + 1} \left[\frac{1}{64} + \frac{1}{4(4+\lambda^2)^2} \right] f\pi^2$$

thus

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$$\begin{aligned}
 F_2 &= -[H_1 f \pi^2 - H_2] \left[-\frac{\sinh \pi}{\pi} g_1 \left(\frac{\sinh \pi}{\pi} + \cosh \pi \right) \right. \\
 &\quad \left. + \left(\frac{\sinh \pi}{\pi} \right)^2 g_2 + \frac{\sinh \pi}{\pi} \cdot \cosh \pi g_1 \right] \frac{1}{\frac{\sinh 2\pi}{2\pi} + 1} \\
 &\quad - [H_3 (f \pi^2)] \left[-g_3 \frac{\sinh 2\pi}{2\pi} \left(\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi \right) + \left(\frac{\sinh 2\pi}{2\pi} \right)^2 g_4 + g_3 \frac{\sinh 2\pi}{2\pi} \cosh 2\pi \right] \\
 &\quad \cdot \frac{1}{\frac{\sinh 4\pi}{4\pi} + 1} \\
 &= -\frac{\left(\frac{\sinh \pi}{\pi} \right)^2}{\frac{\sinh 2\pi}{2\pi} + 1} [H_1 (f \pi^2) - H_2] [g_2 - g_1] \\
 &\quad - \frac{\left(\frac{\sinh 2\pi}{2\pi} \right)^2}{\frac{\sinh 4\pi}{4\pi} + 1} H_3 (f \pi^2)^2 [g_4 - g_3]
 \end{aligned}$$

$$\begin{aligned}
 p_2 - p_1 &= -\frac{\lambda^2}{(1+\lambda^2)^2} \left(\frac{f \pi^2}{8} - 1 \right) + \frac{\lambda^2}{25(1+\lambda^2)^2} (f \pi^2) - \frac{1}{8} (f \pi^2) - \frac{1}{(1+\lambda^2)^2} \left(\frac{f \pi^2}{8} - 1 \right) \\
 &\quad + \frac{1}{200(1+4\lambda^2)^2} (f \pi^2) - \frac{\lambda^2}{(1+\lambda^2)^2} \left(\frac{f \pi^2}{8} - 1 \right) + \frac{\lambda^2}{25(1+4\lambda^2)^2} (f \pi^2) \\
 &= \left(\frac{f \pi^2}{8} - 1 \right) \left(\frac{\lambda^2}{(1+\lambda^2)^2} - 1 \right) + \frac{1}{8} (f \pi^2) + (f \pi^2) \left(\frac{1}{200} - \frac{\lambda^2}{25(1+4\lambda^2)^2} \right) \\
 &= (f \pi^2) \left\{ \frac{1}{200} + \frac{\lambda^2}{8(1+\lambda^2)^2} - \frac{\lambda^2}{25(1+4\lambda^2)^2} \right\} + \left\{ 1 - \frac{\lambda^2}{(1+\lambda^2)^2} \right\}
 \end{aligned}$$

$$f_4 - g_3 = -\frac{\lambda^4}{50(4+\lambda^2)^2} + \frac{1}{32} - \frac{8}{25(4+\lambda^2)^2} - \frac{8\lambda^2}{50(4+\lambda^2)^2}$$

$$= \frac{1}{32} - \frac{1}{50} = \frac{9}{800}$$

$$H_1(\lambda) = \frac{1}{8} \left\{ \frac{1}{2} + \frac{1}{(1+\lambda^2)^2} + \frac{1}{2(1+4\lambda^2)^2} \right\}$$

$$H_2(\lambda) = \frac{1}{(1+\lambda^2)^2}$$

$$H_3(\lambda) = \frac{1}{8} \left\{ \frac{1}{16} + \frac{1}{(4+\lambda^2)^2} \right\}$$

$$H_4(\lambda) = \frac{1}{200} + \frac{\lambda^2}{8(1+\lambda^2)^2} - \frac{\lambda^2}{25(1+4\lambda^2)^2}$$

$$H_5(\lambda) = 1 - \frac{\lambda^2}{(1+\lambda^2)^2}$$

$$F_2 = - \frac{\left(\frac{\sinh 6\pi}{\pi} \right)^2}{\frac{\sinh 2\pi}{2\pi} + 1} \left[H_1(4\pi^2) - H_2 \left[H_4(4\pi^2) + H_5 \right] \right]$$

$$- \frac{9}{800} \frac{\left(\frac{\sinh 2\pi}{2\pi} \right)^2}{\frac{\sinh 4\pi}{4\pi} + 1} H_3(4\pi^2)^2$$

$$F_3 = [H_1(A_1^2) - H_2]^{-2} \times \frac{1}{\left(\frac{\sinh \frac{2\pi}{2\pi}}{2\pi} + 1\right)^2}$$

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$$\begin{aligned} & \left[\frac{1}{4} \left(\frac{\sinh \pi}{\pi} + \cosh \pi \right)^2 \left\{ (1+\lambda^2)^2 \frac{\sinh 2\pi}{2\pi} + (1-\lambda^2)^2 \right\} \right. \\ & - \frac{1}{4} \frac{\sinh \pi}{\pi} \left(\frac{\sinh \pi}{\pi} + \cosh \pi \right) \left\{ (3\lambda^4 + 2\lambda^2 - 1) \frac{\sinh 2\pi}{2\pi} + (1+\lambda^2)^2 \cosh 2\pi - 4(1-\lambda^2)\lambda^2 \right\} \\ & + \frac{1}{4} \left(\frac{\sinh \pi}{\pi} \right)^2 \left\{ \left[\pi^2(1+\lambda^2)^2 + \frac{1}{2}(5\lambda^4 + 2\lambda^2 + 1) \right] \frac{\sinh 2\pi}{2\pi} + \frac{1}{2}(3\lambda^4 + 2\lambda^2 - 1) \cosh 2\pi \right. \\ & \quad \left. - \left[\frac{\pi^2}{3}(1-\lambda^2)^2 + 2(1-2\lambda^4) \right] \right\} \end{aligned}$$

$$+ H_3^2 \left(\frac{A_1^2}{\pi} \right)^2 \times \frac{1}{\left(\frac{\sinh \frac{4\pi}{4\pi}}{4\pi} + 1 \right)^2}$$

$$\begin{aligned} & \left[\frac{1}{4} \left(\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi \right)^2 \left\{ (1+\lambda^2)^2 \frac{\sinh 4\pi}{4\pi} + (1-\lambda^2)^2 \right\} \right. \\ & - \frac{1}{4} \frac{\sinh 2\pi}{2\pi} \left(\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi \right) \left\{ (3\lambda^4 + 2\lambda^2 - 1) \frac{\sinh 4\pi}{4\pi} + (1+\lambda^2)^2 \cosh 4\pi - 4(1-\lambda^2)\lambda^2 \right\} \\ & + \frac{1}{4} \left(\frac{\sinh 2\pi}{2\pi} \right)^2 \left\{ \left[4\pi^2(1+\lambda^2)^2 + \frac{1}{2}(5\lambda^4 + 2\lambda^2 + 1) \right] \frac{\sinh 4\pi}{4\pi} + \frac{1}{2}(3\lambda^4 + 2\lambda^2 - 1) \cosh 4\pi \right. \\ & \quad \left. - \left[\frac{4\pi^2}{3}(1-\lambda^2)^2 + 2(1-2\lambda^4) \right] \right\} \end{aligned}$$

$$\frac{1}{12} \frac{1}{ab} t^2 \int_0^a \int_0^b \left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right)^2 dx dy = \frac{1}{12} \left(\frac{t}{R} \right)^2 \left(\frac{f\pi}{g} \right)^2 \frac{3}{4}$$

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$$\frac{1}{12} \frac{1}{ab} t^2 \int_0^a \int_0^b \left(\frac{\partial^2 \psi}{\partial y^2} \right)^2 dx dy = \frac{1}{12} \left(\frac{t}{R} \right)^2 \left(\frac{f\pi^2 \lambda^2}{g} \right)^2 \frac{3}{4}$$

$$\frac{1}{12} \frac{1}{ab} t^2 \int_0^a \int_0^b \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 dx dy = \frac{1}{12} \left(\frac{t}{R} \right)^2 \left(\frac{f\pi^2 \lambda^2}{g} \right)^2 \frac{1}{4}$$

x 2 !

the binding energy / $(E ab) t \left(\frac{t}{R} \right)$

$$= \frac{1}{12} \left(\frac{t}{R} \right)^2 \left(\frac{f}{g} \right)^2 \pi^4 \left[\frac{3}{4} (1 + \lambda^4) + \frac{1}{2} \lambda^2 \right]$$

$$= \left(\frac{t}{R} \right)^2 \left(\frac{f}{g} \right)^2 \pi^4 \left[\frac{1}{16} (1 + \lambda^4) + \frac{1}{24} \lambda^2 \right]$$

$$\frac{1}{2} \left(\frac{\partial \psi}{\partial y} \right)^2 = \frac{1}{2} \left[\frac{f\lambda\pi}{g} (1 + \cos \frac{\pi x}{a}) \sin \frac{\lambda\pi y}{g} \right]^2 \left(\frac{t}{R} \right)^2$$

$$= \left(\frac{t}{R} \right)^2 \left(\frac{f\lambda^2}{g} \right)^2 \left[\frac{f\pi^2}{16} \frac{1}{4} (3 + 2 \cos \frac{\pi x}{a} + \cos^2 \frac{\pi x}{a}) (1 - \cos 2\pi \frac{\lambda y}{g}) \right]$$

$$\frac{\partial V}{\partial y} = \left(\frac{t}{R} \right)^2 \left(\frac{f\lambda^2}{g} \right)^2 \left[-\frac{3}{4} \frac{f\pi^2}{16} + \dots \right] - \frac{\sigma}{E}$$

$$\left(\frac{V}{b} \right) = \left(\frac{t}{R} \right)^2 \left(\frac{f\lambda^2}{g} \right)^2 \left[-\frac{3}{4} \frac{f\pi^2}{16} + \dots \right] - \frac{\sigma}{E}$$

$$\begin{aligned}
 2 \times \frac{\Delta f^2}{E a b t} &= -\frac{\sigma}{E} \left(\frac{a}{R}\right)^2 \left(\frac{f \lambda^2}{\rho}\right) \frac{3}{32} (f \pi) - 2 \left(\frac{\sigma}{E}\right)^2 \\
 &= -\left(\frac{a}{R}\right)^2 \left(\frac{f}{\rho}\right)^2 \lambda^2 \frac{3}{4} \pi^2 \frac{\sigma}{E} - 2 \left(\frac{\sigma}{E}\right)^2
 \end{aligned}$$

In limiting case of $\lambda \ll 1$

$$\frac{3}{2} \pi^2 \frac{\sigma}{E} = \frac{1}{\lambda^2} \left(\frac{a}{R}\right)^2 \left\{ \frac{17}{1048} \pi^2 f^2 - \frac{3}{16} \pi^2 f + 1 \right\} + \frac{1}{8} \pi^2 \frac{\left(\frac{a}{R}\right)^2}{\left(\frac{\rho}{R}\right)^2} \frac{1}{\lambda^2}$$

$$\lambda^2 K = f^2 \left\{ \frac{17}{3072} \pi^2 f^2 - \frac{1}{8} f + \frac{2}{3\pi} \right\} + \frac{1}{12} \frac{\pi^2}{f^2}$$

$$= \frac{\pi^2}{f^2} \left\{ \frac{17}{768} \left(\frac{f}{t}\right)^2 + \frac{1}{12} \right\} - \frac{1}{4} \left(\frac{f}{t}\right) + \frac{2}{3} \frac{f}{\pi^2}$$

$$= 2 \left\{ \frac{17}{1152} \left(\frac{f}{t}\right)^2 + \frac{1}{18} \right\} - \frac{1}{4} \left(\frac{f}{t}\right)$$

$$\left(\frac{1}{64} - \frac{17}{1152} \right) \left(\frac{f}{t}\right)^2 = \frac{1}{18}$$

$$\left(\frac{f}{t}\right)^2 = \frac{1}{18} \times 1152 \quad \therefore \left(\frac{f}{t}\right) = 8$$

$$\frac{u}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left[1 - \left(\frac{x}{a} \right)^2 - \frac{f}{4} \left(1 + \cos \frac{\pi x}{a} \right) \left(1 + \cos \frac{\pi y}{b} \right) \right]$$

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$$\frac{u_0}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left[1 - \left(\frac{x}{a} \right)^2 \right]$$

$$\sigma_x = E \left\{ \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 - \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right\}$$

$$\sigma_y = E \left\{ \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right\}$$

$$\tau_{xy} = E \left\{ \frac{1}{2} \frac{\partial v}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right) \left(\frac{\partial w}{\partial y} \right) \right\}$$

By using the equilibrium equation $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$, we have

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= - \left\{ \frac{1}{2} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial y} + \frac{1}{2} \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y^2} \right\} \\ &= - \left\{ \frac{1}{2} \frac{\partial w}{\partial y} \left(\frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \right) - \frac{1}{2} \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right\} \end{aligned}$$

We have

$$\frac{\partial w}{\partial x} = \left(\frac{a}{R} \right) \left[- \left(\frac{x}{a} \right) + \frac{f}{8} \pi \sin \frac{\pi x}{a} \left(1 + \cos \frac{\pi y}{b} \right) \right]$$

$$\frac{\partial w}{\partial y} = \left(\frac{a}{R} \right) \left[\frac{f}{8} \pi \left(\frac{a}{b} \right) \left(1 + \cos \frac{\pi x}{a} \right) \sin \frac{\pi y}{b} \right]$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{R} \left[-1 + \frac{f}{8} \pi^2 \cos \frac{\pi x}{a} \left(1 + \cos \frac{\pi y}{b} \right) \right]$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{R} \left[\frac{f}{8} \pi^2 \left(\frac{a}{b} \right)^2 \left(1 + \cos \frac{\pi x}{a} \right) \cos \frac{\pi y}{b} \right]$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{R} \left[- \frac{f}{8} \pi^2 \left(\frac{a}{b} \right) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \right]$$

$$R\left(\frac{\partial^2 U}{\partial x^2} + 2\frac{\partial^2 U}{\partial y^2}\right) = -\left(\frac{a}{R}\right) \left[\frac{f}{8} \pi \left(\frac{a}{b}\right)^2 (1 + \cos \frac{2\pi x}{a}) \sin \frac{\pi y}{b} \right] \left[\frac{f}{8} \pi^2 \cos \frac{2\pi x}{a} (1 + \cos \frac{2\pi y}{b}) + \frac{f}{8} \pi^2 \left(\frac{a}{b}\right)^2 (1 + \cos \frac{2\pi x}{a}) \cos \frac{2\pi y}{b} - 1 \right]$$

$$- \left[\frac{f}{8} \pi \sin \frac{2\pi x}{a} (1 + \cos \frac{2\pi y}{b}) - \left(\frac{a}{R}\right) \left[\frac{f}{8} \pi^2 \left(\frac{a}{b}\right)^2 \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{b} \right] \right]$$

$$= -\left(\frac{a}{R}\right) \left[\left(\frac{f}{8}\right)^2 \pi^3 \left(\frac{a}{b}\right)^2 \frac{1}{4} \left(1 + 2 \cos \frac{2\pi x}{a} + \cos^2 \frac{2\pi x}{a} \right) (2 \sin \frac{\pi y}{b} + \sin \frac{2\pi y}{b}) \right]$$

$$+ \left(\frac{f}{8}\right)^2 \pi^3 \left(\frac{a}{b}\right)^3 \frac{1}{4} \left(3 + 4 \cos \frac{2\pi x}{a} + \cos^2 \frac{2\pi x}{a} \right) \sin \frac{2\pi y}{b} - \frac{f}{8} \pi \left(\frac{a}{b}\right)^2 (1 + \cos \frac{2\pi x}{a}) \sin \frac{\pi y}{b}$$

$$- \left(\frac{f}{8}\right)^2 \pi^3 \left(\frac{a}{b}\right)^2 \frac{1}{4} \left(1 - \cos \frac{2\pi x}{a} \right) \left(2 \sin \frac{\pi y}{b} + \sin \frac{2\pi y}{b} \right) + \frac{f}{8} \pi^2 \left(\frac{a}{b}\right)^2 \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} \right]$$

$$= -\left(\frac{a}{R}\right) \left(\frac{f}{8}\right) \pi \left(\frac{a}{b}\right)^2 \left[\frac{f}{4} \left(\frac{f}{8}\right)^2 \pi^2 \left(4 \cos \frac{2\pi x}{a} \sin \frac{\pi y}{b} + 4 \cos \frac{2\pi x}{a} \sin \frac{2\pi y}{b} + 2 \cos \frac{2\pi x}{a} \sin \frac{2\pi y}{b} + 2 \cos \frac{2\pi x}{a} \sin \frac{2\pi y}{b} \right) \right. \\ \left. + \frac{f}{4} \left(\frac{f}{8}\right)^2 \pi^2 \left(\frac{a}{b}\right)^2 \left(3 \sin \frac{2\pi y}{b} + 4 \cos \frac{2\pi x}{a} \sin \frac{2\pi y}{b} + \cos \frac{2\pi x}{a} \sin \frac{2\pi y}{b} \right) \right. \\ \left. - \left\{ 1 + \cos \frac{2\pi x}{a} - \left(\frac{2\pi x}{a}\right) \sin \frac{2\pi x}{a} \right\} \sin \frac{\pi y}{b} \right]$$

$$= -\left(\frac{a}{R}\right) \left(\frac{f}{8}\right) \pi \left(\frac{a}{b}\right)^2 \left[\frac{f}{4} \left(\frac{f}{8}\right)^2 \pi^2 \left\{ 3 \left(\frac{a}{b}\right)^2 \sin \frac{2\pi y}{b} + 4 \cos \frac{2\pi x}{a} \sin \frac{\pi y}{b} + 4 \cos \frac{2\pi x}{a} \sin \frac{2\pi y}{b} + (2 + 4 \left(\frac{a}{b}\right)^2) \cos \frac{2\pi x}{a} \sin \frac{2\pi y}{b} \right. \right. \\ \left. \left. + (2 + \left(\frac{a}{b}\right)^2) \cos \frac{2\pi x}{a} \sin \frac{2\pi y}{b} \right\} \right. \\ \left. - \left\{ \sin \frac{\pi y}{b} + \cos \frac{2\pi x}{a} \sin \frac{\pi y}{b} - \left(\frac{2\pi x}{a}\right) \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} \right\} \right]$$

Consider $\mathcal{R} \left(\frac{\partial^2 \chi}{\partial x^2} + 2 \frac{\partial^2 \chi}{\partial y^2} \right) = \left(\frac{\partial \chi}{\partial a} \right) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$

Let $\chi = X \sin \frac{\pi y}{b}$

$$\mathcal{R} \left[\frac{d^2 X}{dx^2} - 2 \left(\frac{\pi}{b} \right)^2 X \right] = \left(\frac{\partial \chi}{\partial a} \right) \sin \frac{\pi x}{a}$$

Let $X = A \cos \frac{\pi x}{a} + B \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a}$

$$\frac{dX}{dx} = -\left(\frac{\pi}{a} \right) A \sin \frac{\pi x}{a} + \left(\frac{\pi}{a} \right) B \sin \frac{\pi x}{a} + \left(\frac{\pi^2}{a} \right) B \cos \frac{\pi x}{a}$$

$$\frac{d^2 X}{dx^2} = -\left(\frac{\pi}{a} \right)^2 (A - 2B) \cos \frac{\pi x}{a} - \left(\frac{\pi^2}{a} \right) B \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a}$$

$$\mathcal{R} \left[\left\{ -\left(\frac{\pi}{a} \right)^2 (A - 2B) - 2 \left(\frac{\pi}{b} \right)^2 A \right\} \cos \frac{\pi x}{a} - \left\{ \left(\frac{\pi}{a} \right)^2 B + 2 \left(\frac{\pi}{b} \right)^2 B \right\} \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a} \right] = \left(\frac{\partial \chi}{\partial a} \right) \sin \frac{\pi x}{a}$$

$$\therefore B = - \frac{1}{\left(\frac{\pi}{a} \right)^2 + 2 \left(\frac{\pi}{b} \right)^2} \cdot \frac{1}{a}$$

$$\left[\left(\frac{\pi}{a} \right)^2 + 2 \left(\frac{\pi}{b} \right)^2 \right] A = 2 \left(\frac{\pi}{a} \right)^2 B,$$

$$A = \frac{\left(\frac{\pi}{a} \right)^2}{\left(\frac{\pi}{a} \right)^2 + 2 \left(\frac{\pi}{b} \right)^2} B = - \frac{\left(\frac{\pi}{a} \right)^2}{\left[\left(\frac{\pi}{a} \right)^2 + 2 \left(\frac{\pi}{b} \right)^2 \right]^2} \cdot \frac{1}{a}$$

The particular integral:

$$\begin{aligned}
 \frac{v}{R} = & + \left(\frac{a}{R}\right) \left(\frac{1}{8}\right) \pi \left(\frac{a}{8}\right) \frac{1}{R^2} \left[\frac{3}{4} \left(\frac{1}{8}\right) \pi^2 \right] \left\{ \frac{3}{2} \left(\frac{1}{8}\right)^2 \sin \frac{2\pi x}{8} + \frac{4}{\left(\frac{1}{8}\right)^2 + 2 \left(\frac{1}{8}\right)^2} \cos \frac{2\pi x}{8} \sin \frac{\pi x}{8} + \frac{1}{\left(\frac{1}{8}\right)^2 + 2 \left(\frac{1}{8}\right)^2} \cos \frac{2\pi x}{8} \sin \frac{\pi x}{8} \right. \\
 & + \frac{2 + 4 \left(\frac{9}{8}\right)^2}{\left(\frac{1}{8}\right)^2 + 2 \left(\frac{27}{8}\right)^2} \cos \frac{\pi x}{8} \sin \frac{\pi x}{8} + \frac{2 + \left(\frac{9}{8}\right)^2}{\left(\frac{27}{8}\right)^2 + 2 \left(\frac{27}{8}\right)^2} \cos \frac{2\pi x}{8} \sin \frac{2\pi x}{8} \left. \right\} \\
 & - \left\{ \frac{1}{2 \left(\frac{1}{8}\right)^2} \sin \frac{\pi x}{8} + \frac{1/\left(\frac{1}{8}\right)}{\left[\left(\frac{1}{8}\right)^2 + 2 \left(\frac{1}{8}\right)^2\right]} \cos \frac{\pi x}{8} - \frac{2 \left(\frac{1}{8}\right)^2}{\left[\left(\frac{1}{8}\right)^2 + 2 \left(\frac{1}{8}\right)^2\right]} \cos \frac{2\pi x}{8} \sin \frac{\pi x}{8} \right. \\
 & \left. \left. - \frac{1}{\left(\frac{1}{8}\right)^2 + 2 \left(\frac{1}{8}\right)^2} \left(\frac{1}{8}\right) \sin \left(\frac{\pi x}{8}\right) \sin \frac{2\pi x}{8} \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 \frac{v}{R} = & \left(\frac{a}{R}\right) \left(\frac{1}{8}\right) \pi \left(\frac{a}{8}\right) \frac{1}{R^2} \left[\frac{3}{4} \left(\frac{1}{8}\right) \pi^2 \right] \left\{ \frac{3}{8} \sin \frac{2\pi x}{8} + \frac{4}{(1+2\lambda^2)} \cos \frac{2\pi x}{8} \sin \frac{\pi x}{8} + \frac{4}{(4+2\lambda^2)} \cos \frac{2\pi x}{8} \sin \frac{\pi x}{8} \right. \\
 & + \frac{2+4\lambda^2}{(1+8\lambda^2)} \cos \frac{\pi x}{8} \sin \frac{2\pi x}{8} + \frac{2+\lambda^2}{4+8\lambda^2} \cos \frac{2\pi x}{8} \sin \frac{2\pi x}{8} \left. \right\} \\
 & - \frac{1}{R^2} \left[\frac{1}{2\lambda^2} \sin \frac{\pi x}{8} + \frac{1}{(1+2\lambda^2)} \cos \frac{\pi x}{8} \sin \frac{\pi x}{8} - \frac{1}{(1+2\lambda^2)} \left(\frac{1}{8}\right) \sin \left(\frac{\pi x}{8}\right) \sin \frac{2\pi x}{8} \right] \\
 & + \frac{1}{R^2} \left[\frac{1}{2\lambda^2} \sin \frac{\pi x}{8} + \frac{1}{4} a_1 \cos \frac{\sqrt{2} \pi x}{a} \sin \frac{\pi x}{8} \right]
 \end{aligned}$$

Add the complementary function.

$$A_0 \left(\frac{\pi x}{b} \right) + A_1 \cosh \sqrt{2} \frac{\pi x}{b} \sin \frac{\pi x}{b}$$

$$\begin{aligned} \frac{\partial \psi}{\partial y} = & \left(\frac{a}{R} \right)^2 \left(\frac{1}{f} \right) \lambda^2 \left[\frac{A_1}{32} \left\{ \frac{3}{4} \cos \frac{2\pi y}{b} + \frac{4}{(1+\lambda^2)} \cos \frac{\pi y}{b} + \frac{4}{(4+2\lambda^2)} \cos \frac{2\pi y}{b} \right. \right. \\ & + \frac{4(1+2\lambda^2)}{(1+\lambda^2)^2} \cos \frac{\pi x}{a} \cos \frac{2\pi y}{b} + \frac{2+\lambda^2}{9(2+2\lambda^2)} \cos \frac{2\pi x}{a} \cos \frac{2\pi y}{b} \Big\} \\ & - \left\{ \frac{1}{2\lambda^2} \cos \frac{\pi y}{b} + \frac{1}{(1+\lambda^2)^2} \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} - \frac{1}{(1+\lambda^2)} \left(\frac{\pi x}{a} \right) \sin \left(\frac{\pi x}{a} \right) \cos \frac{\pi y}{b} \right\} \\ & \left. + a_0 + a_1 \cosh \sqrt{2} \lambda \frac{\pi x}{a} \cos \frac{\pi y}{b} \right] \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial \psi}{\partial y} \right)_{y=\pm b} = & \left(\frac{a}{R} \right)^2 \left(\frac{1}{f} \right) \lambda^2 \left[\frac{A_1}{32} \left\{ \frac{3}{4} - \frac{4}{(1+\lambda^2)} \cos \frac{\pi x}{a} - \frac{4}{(4+2\lambda^2)} \cos \frac{2\pi x}{a} + \frac{4(1+2\lambda^2)}{1+8\lambda^2} \cos \frac{\pi x}{a} \right. \right. \\ & + \frac{2+\lambda^2}{2(2+2\lambda^2)} \cos \frac{2\pi x}{a} \Big\} + \left\{ \frac{1}{2\lambda^2} + \frac{1}{(1+\lambda^2)^2} \cos \frac{\pi x}{a} - \frac{1}{(1+\lambda^2)} \left(\frac{\pi x}{a} \right) \sin \left(\frac{\pi x}{a} \right) \right\} \\ & \left. + a_0 - a_1 \cosh \sqrt{2} \lambda \frac{\pi x}{a} \right] \end{aligned}$$

$$\text{Average stress} = \left(\frac{Q}{R}\right)^2 \left(\frac{1}{g}\right) \lambda^2 \left[\frac{3}{12g} g^2 + \frac{1}{2\lambda^2} + a_0 \right] = - \frac{Q}{E}$$

The decrease in potential

$$\frac{\Delta \phi}{abEt} = \left[\left(\frac{Q}{R}\right)^2 \left(\frac{1}{g}\right) \lambda^2 a_0 \left[\frac{3}{12g} g^2 + \frac{1}{2\lambda^2} + a_0 \right] \right]$$

Decrease in Potential

$$\frac{\Delta \phi}{abEt} = + \frac{Q}{E} \left\{ \frac{Q}{E} + \left(\frac{Q}{R}\right)^2 \left(\frac{1}{g}\right) \lambda^2 \left(\frac{3}{12g} g^2 + \frac{1}{2\lambda^2} \right) \right\}$$

$$\frac{V}{R} = \left(\frac{Q}{R}\right)^3 \left(\frac{1}{g}\right) \pi \lambda \left[\frac{1}{32} \left\{ \frac{3}{g} \sin \frac{2\pi x}{b} + \frac{1}{(1+2\lambda^2)} \cos \frac{\pi x}{a} \sin \frac{\pi x}{b} + \frac{2}{(2+\lambda^2)} \cos \frac{2\pi x}{a} \sin \frac{\pi x}{b} \right\} \right]$$

$$+ \frac{2(1+2\lambda^2)}{(1+8\lambda^2)} \cos \frac{\pi x}{a} \sin \frac{2\pi x}{b} + \frac{2+2^2}{4(1+2\lambda^2)} \cos \frac{2\pi x}{a} \sin \frac{2\pi x}{b} \left\{ \right.$$

$$\left. - \frac{1}{\pi} \left[\frac{1}{2\lambda^2} \sin \frac{\pi x}{b} + \frac{1}{(1+2\lambda^2)^2} \cos \frac{\pi x}{a} \sin \frac{\pi x}{b} - \frac{1}{(1+2\lambda^2)} \left(\frac{\pi x}{a}\right) \sin \left(\frac{\pi x}{a}\right) \sin \frac{\pi x}{b} \right] \right\}$$

$$- \left(\frac{3}{12g} + \frac{1}{2\lambda^2} \right) \left(\frac{\pi x}{b} \right) + \frac{1}{b} a, \cos \frac{\sqrt{2} \lambda \pi x}{a} \sin \frac{\pi x}{b} \left] - \frac{Q}{E} \left(\frac{1}{R} \right)$$

$$\begin{aligned} \frac{\partial V}{\partial x} = & \cdot \left(\frac{a}{b}\right)^2 \left(\frac{f}{g}\right) \lambda \left[\frac{fx}{32} \left(-\frac{6}{(1+2\lambda^2)} \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} - \frac{6}{(2+\lambda^2)} \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} - \frac{2(1+2\lambda^2)}{(1+8\lambda^2)} \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} \right. \right. \\ & \left. \left. - \frac{2+\lambda^2}{2(1+2\lambda^2)} \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{b} \right\} + \left\{ \frac{2(1+\lambda^2)}{(1+2\lambda^2)^2} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} + \frac{1}{(1+2\lambda^2)} \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} \right. \right. \\ & \left. \left. + \frac{1}{(1+2\lambda^2)} \left(\frac{\pi}{a}\right) \cos \left(\frac{\pi x}{a}\right) \sin \frac{\pi y}{b} \right\} \right] \end{aligned}$$

$$+ \sqrt{2} \lambda a_1 \sinh \frac{\sqrt{2} \lambda \pi x}{a} \sin \frac{\pi y}{b} \Big]$$

$$\left(\frac{\partial V}{\partial x}\right) = 0 \quad \text{when} \quad x = \pm a,$$

$$\frac{1}{(1+2\lambda^2)} \pi = \sqrt{2} \lambda a_1 \sinh(\sqrt{2} \lambda \pi)$$

$$a_1 = \frac{\pi}{(1+2\lambda^2) \sqrt{2} \lambda \sinh(\sqrt{2} \lambda \pi)}$$

$$\begin{aligned}
 \frac{\partial \psi}{\partial y} = & \left(\frac{q}{R}\right)^2 \left(\frac{1}{8}\right) \lambda^2 \left[\frac{f \pi^2}{32} \left\{ \frac{3}{4} \cos \frac{2\pi x}{b} + \frac{4}{(1+\lambda^2)} \cos \frac{\pi x}{a} \cos \frac{\pi x}{b} + \frac{2}{(2+\lambda^2)} \cos \frac{2\pi x}{a} \cos \frac{\pi x}{b} \right. \right. \\
 & + \left. \frac{4(1+\lambda^2)}{(1+\lambda^2)^2} \cos \frac{\pi x}{a} \cos \frac{2\pi x}{b} + \frac{2+\lambda^2}{2(1+\lambda^2)} \cos \frac{2\pi x}{a} \cos \frac{2\pi x}{b} \right\} \\
 & - \left[\frac{1}{2\lambda^2} \cos \frac{\pi x}{b} + \frac{1}{(1+2\lambda^2)^2} \cos \frac{\pi x}{a} \cos \frac{\pi x}{b} - \frac{1}{(1+2\lambda^2)} \left(\frac{\pi x}{a} \right) \sin \left(\frac{\pi x}{a} \right) \cos \frac{\pi x}{b} \right] \\
 & - \left(\frac{3}{128} f \lambda^2 + \frac{1}{2\lambda^2} \right) + a_1 \cos \left[\frac{\sqrt{2} \lambda \pi x}{a} \cos \frac{\pi y}{b} \right] - \frac{\sigma}{E}
 \end{aligned}$$

$$\frac{1}{2} \left(\frac{\partial \omega}{\partial y} \right)^2 = \left(\frac{q}{R} \right)^2 \left(\frac{1}{8} \right) \lambda^2 \left[\frac{3}{64} f \lambda^2 + \frac{1}{16} f \lambda^2 \cos \frac{\pi x}{a} + \frac{1}{64} f \lambda^2 \cos \frac{2\pi x}{a} - \frac{3}{64} f \lambda^2 \cos \frac{2\pi x}{b} - \frac{1}{16} f \lambda^2 \cos \frac{\pi x}{a} \cos \frac{2\pi x}{b} - \frac{1}{64} f \lambda^2 \cos \frac{2\pi x}{a} \cos \frac{2\pi x}{b} \right]$$

$$\begin{aligned}
 \frac{\partial \psi}{\partial x} = & \left(\frac{q}{R}\right)^2 \frac{f}{8} \lambda^2 \left[\left(\frac{3}{128} f \lambda^2 - \frac{1}{\lambda^2} \right) + \left(\frac{1}{16} f \lambda^2 \cos \frac{\pi x}{a} + \frac{1}{64} f \lambda^2 \cos \frac{2\pi x}{a} \right) \right. \\
 & + \left\{ \frac{1}{8(1+2\lambda^2)} f \lambda^2 \cos \frac{\pi x}{a} + \frac{1}{16(2+\lambda^2)} f \lambda^2 \cos \frac{2\pi x}{a} - \frac{1}{2\lambda^2} - \frac{1}{(1+2\lambda^2)^2} \cos \frac{\pi x}{a} + \frac{1}{(1+2\lambda^2)} \frac{\pi x}{a} \sin \frac{\pi x}{a} + a_1 \cos \left[\frac{\sqrt{2} \lambda \pi x}{a} \right] \cos \frac{\pi y}{b} \right\} \\
 & + \left\{ -\frac{3}{128} f \lambda^2 + \frac{1-4\lambda^2}{16(1+\lambda^2)} f \lambda^2 \frac{\pi x}{a} + \frac{1-\lambda^2}{64(1+2\lambda^2)} f \lambda^2 \cos \frac{2\pi x}{a} \right\} \cos \frac{2\pi y}{b} - \frac{\sigma}{E}
 \end{aligned}$$

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$$\frac{1}{a} \int_0^a \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a} \cos \frac{\pi x}{a} dx = \frac{1}{\pi a} \int_0^{2\pi} \theta \sin \theta d\theta = \frac{1}{\pi} \left[\sin \theta - \theta \cos \theta \right]_0^{2\pi} = -\frac{1}{8\pi} 2\pi = -\frac{1}{4}$$

$$\begin{aligned} \frac{1}{a} \int_0^a \cos \frac{\pi x}{a} \cosh(\sqrt{2}x) \frac{\pi x}{a} dx &= \frac{1}{2\pi} \int_0^a \left[\cosh \frac{\pi x}{a} (\sqrt{2}x + i) + \cosh \frac{\pi x}{a} (\sqrt{2}x - i) \right] d\left(\frac{\pi x}{a}\right) \\ &= \frac{1}{2\pi} \int \frac{1}{\sqrt{2}x + i} \sinh \frac{\pi x}{a} (\sqrt{2}x + i) + \frac{1}{\sqrt{2}x - i} \sinh \frac{\pi x}{a} (\sqrt{2}x - i) dx = -\frac{1}{\pi} \frac{\sqrt{2}x}{2x^2 + 1} \sinh \sqrt{2}x \pi \end{aligned}$$

$$\begin{aligned} \frac{1}{a} \int_0^a \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a} \cos \frac{2\pi x}{a} dx &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\pi x}{a} \right) \left[\sin \frac{3\pi x}{a} - \sin \frac{\pi x}{a} \right] d\left(\frac{\pi x}{a}\right) \\ &= \frac{1}{2\pi} \left[\frac{1}{9} (\sin \theta - \theta \cos \theta) \right]_0^{3\pi} - \left[\sin \theta - \theta \cos \theta \right]_0^{\pi} = \frac{1}{2} \left(\frac{1}{3} - 1 \right) = -\frac{1}{3} \end{aligned}$$

$$\frac{1}{a} \int_0^a \cos \frac{2\pi x}{a} \cosh(\sqrt{2}x) \frac{\pi x}{a} dx = \frac{1}{\pi} \frac{\sqrt{2}x \sinh \sqrt{2}x \pi}{2x^2 + 4}$$

$$\frac{1}{a} \int_0^a \frac{\pi x}{a} \sin \frac{\pi x}{a} dx = 1$$

$$\frac{1}{a} \int_0^a \cosh \sqrt{2}x \frac{\pi x}{a} dx = \frac{\sinh \sqrt{2}x \pi}{\sqrt{2}x \pi}$$

$$\begin{aligned} \frac{1}{a} \int_0^a \left(\frac{\pi x}{a} \right)^2 \sin \frac{\pi x}{a} \frac{\pi x}{a} dx &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\pi x}{a} \right)^2 \left[\cos \frac{2\pi x}{a} \right] d\left(\frac{\pi x}{a}\right) = \frac{1}{2\pi} \left[\frac{\pi^3}{3} - \frac{1}{2} \{ 2\theta \cos \theta + (\theta^2 - 2) \sin \theta \} \right]_0^{2\pi} \\ &= \frac{\pi^3}{6} - \frac{1}{16\pi} (4\pi) = \frac{\pi^2}{6} - \frac{1}{4} \end{aligned}$$

$$\begin{aligned}
\frac{1}{a} \int_0^a \frac{\frac{\pi x}{a} \sinh \frac{\pi x}{a} \cosh(\sqrt{2}\lambda) \frac{\pi x}{a}}{(\sqrt{\lambda} \lambda + i)^2} dx &= \frac{1}{2\pi i} \int_0^a \left[\frac{\pi x}{a} \sinh(\sqrt{2}\lambda + i) - \frac{\pi x}{a} \sinh(\sqrt{2}\lambda - i) \right] \frac{1}{(\sqrt{\lambda} \lambda + i)^2} dx \\
&= \frac{1}{2\pi i} \left[\frac{1}{(\sqrt{\lambda} \lambda + i)^2} \left[\frac{\pi x}{a} (\sqrt{2}\lambda + i) \cosh \frac{\pi x}{a} (\sqrt{2}\lambda + i) - \sinh \frac{\pi x}{a} (\sqrt{2}\lambda + i) \right] \right. \\
&\quad \left. - \frac{1}{(\sqrt{\lambda} \lambda - i)^2} \left[\frac{\pi x}{a} (\sqrt{2}\lambda - i) \cosh \frac{\pi x}{a} (\sqrt{2}\lambda - i) - \sinh \frac{\pi x}{a} (\sqrt{2}\lambda - i) \right] \right] \Big|_0^a \\
&= \frac{1}{2\pi i} \left[\frac{\pi x}{a} \left\{ \frac{\cosh \frac{\pi x}{a} (\sqrt{2}\lambda + i)}{\sqrt{2}\lambda + i} - \frac{\cosh \frac{\pi x}{a} (\sqrt{2}\lambda - i)}{\sqrt{2}\lambda - i} \right\} - \left\{ \frac{\sinh \frac{\pi x}{a} (\sqrt{2}\lambda + i)}{(\sqrt{2}\lambda + i)^2} - \frac{\sinh \frac{\pi x}{a} (\sqrt{2}\lambda - i)}{(\sqrt{2}\lambda - i)^2} \right\} \right] \Big|_0^a \\
&= \frac{\cosh \sqrt{2}\lambda \pi}{2\lambda^2 + 1} - \frac{2\sqrt{2}\lambda \sinh(\sqrt{2}\lambda \pi)}{\pi[(\lambda^2 - 1) + \lambda^2]} = \frac{\cosh \sqrt{2}\lambda \pi}{2\lambda^2 + 1} - \frac{2\sqrt{2}\lambda \sinh \sqrt{2}\lambda \pi}{\pi(2\lambda^2 + 1)}
\end{aligned}$$

$$\frac{1}{a} \int_0^a \cosh^2(\sqrt{2}\lambda) \frac{\pi x}{a} dx = \frac{1}{2a} \int_0^a [1 + \cosh(2\sqrt{2}\lambda) \frac{\pi x}{a}] dx = \frac{1}{2} + \frac{1}{4\sqrt{2}\lambda \pi} \sinh 2\sqrt{2}\lambda \pi$$

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$$\begin{aligned}
& \frac{1}{gE^2 ab} \int_0^a \int_0^b \sigma_y^2 dx dy = \frac{1}{2} \left(\frac{\sigma}{E} \right)^2 - \frac{12}{K} \frac{1}{8} \frac{\sigma}{E} \left[\left(\frac{3}{28} f_1^2 - \frac{1}{24} \right) \right] \\
& + \left(\frac{9}{8} \right) \left(\frac{f_1^2}{8} \right)^2 \left[\frac{1}{2} \left(\frac{3}{28} f_1^2 - \frac{1}{24} \right)^2 + \frac{1}{4} \left\{ \left(\frac{f_1^2}{16} \right)^2 + \left(\frac{f_1^2}{64} \right)^2 \right\} \right] \\
& + \frac{1}{8} \left\{ 2 \left(\frac{3}{28} f_1^2 \right)^2 + \left(\frac{1-4\lambda^2}{16(1+f_1^2)} \right)^2 + \left(\frac{1-\lambda^2}{64(1+\lambda^2)} f_1^2 \right)^2 + \left(\frac{f_1^2}{8(1+\lambda^2)} - \frac{1}{16(2+\lambda^2)} \right)^2 + \left(\frac{f_1^2}{16(2+\lambda^2)} + 2 \left(\frac{1}{24} \right)^2 \right) \right\} \\
& + \frac{1}{40} \int_0^a \int_0^b \left(\frac{f_1^2}{8(1+\lambda^2)} - \frac{1}{(1+\lambda^2)^2} \right) \left(\frac{1}{(1+\lambda^2)} \frac{\pi}{a} \sin \frac{\pi x}{a} + a_1 \cosh \frac{\sqrt{2} \lambda \pi x}{a} \right) \cos \frac{\pi x}{a} dx \\
& + 2 \int_0^a \frac{f_1^2}{16(2+\lambda^2)} \left(\frac{1}{(1+\lambda^2)} \frac{\pi}{a} \sin \frac{\pi x}{a} + a_1 \cosh \frac{\sqrt{2} \lambda \pi x}{a} \right) \cos \frac{2\pi x}{a} dx \\
& - \frac{1}{24} \int_0^a \left(\frac{1}{(1+\lambda^2)} \frac{\pi}{a} \sin \frac{\pi x}{a} + a_1 \cosh \frac{\sqrt{2} \lambda \pi x}{a} \right) dx \\
& + \frac{1}{4} \int_0^a \left(\frac{1}{(1+\lambda^2)} \frac{\pi}{a} \sin \frac{\pi x}{a} + a_1 \cosh \frac{\sqrt{2} \lambda \pi x}{a} \right)^2 dx \Bigg]
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2E_{ab}} \int_0^a \int_0^b \phi_0^2 dx dy &= \frac{1}{2} \left(\frac{a}{E} \right)^2 + \left(\frac{a}{K} \right)^2 \frac{f}{8} \lambda^2 \frac{E}{f} \left(\frac{1}{2\lambda^2} - \frac{3}{128} \frac{a^2}{b^2} \right) \\
&+ \left(\frac{a}{K} \right)^4 \left(\frac{1}{8} \frac{\lambda^2}{b^2} \right)^2 \left[\frac{1}{8} \left(\frac{3}{64} \frac{a^2}{b^2} - \frac{1}{\lambda^2} \right)^2 + \frac{\frac{3}{64} \frac{a^2}{b^2}}{65536} \left(\frac{a^2}{b^2} \right)^2 + \frac{\frac{1}{2048} \frac{a^2}{b^2}}{(1+2\lambda^2)^2} \left(\frac{a^2}{b^2} \right)^2 + \frac{1}{32} \frac{\frac{1}{(1+2\lambda^2)^2} \left(\frac{a^2}{b^2} \right)^2}{(1+2\lambda^2)^2} \right. \\
&- \frac{1}{32} \frac{\frac{1}{(1+2\lambda^2)^3} \left(\frac{a^2}{b^2} \right)}{f(1+2\lambda^2)^4} + \frac{1}{2048} \frac{\frac{1}{(1+2\lambda^2)^2} \left(\frac{a^2}{b^2} \right)}{(1+2\lambda^2)^2} + \frac{1}{2} \left(\frac{1}{f(1+2\lambda^2)^2} - \frac{1}{4(1+2\lambda^2)^2} \right) \left. - \frac{1}{4} \frac{1}{(1+2\lambda^2)^2} \right\} \\
&- \frac{1}{(1+2\lambda^2)^2} \left\{ + \frac{1}{2} \frac{\frac{1}{16(1+2\lambda^2)^2}}{f(1+2\lambda^2)^2} \left\{ - \frac{1}{3(1+2\lambda^2)^2} + \frac{1}{2} \frac{1}{(1+2\lambda^2)^2(2+\lambda^2)} \right\} - \frac{1}{4\lambda^2} \frac{1}{(1+2\lambda^2)^2} - \frac{1}{4\lambda^2} \frac{1}{(1+2\lambda^2)^2} \right. \\
&\left. + \frac{1}{4} \frac{1}{(1+2\lambda^2)^2} \left(\frac{\pi^2}{6} - \frac{1}{4} \right) + \frac{1}{2} \frac{1}{(1+2\lambda^2)^2} \frac{\pi}{\sqrt{2}\lambda} \operatorname{erfc} \sqrt{2}\lambda\pi - \frac{1}{2} \frac{1}{(1+2\lambda^2)^2} + \frac{1}{4} \frac{\pi^2}{(1+2\lambda^2)^2} \lambda^2 (\cos 2\pi\lambda\pi - 1) \right\}
\end{aligned}$$

$$\begin{aligned}
\frac{\phi_1}{E} &= \frac{1}{2} \left(\frac{a}{b\lambda} \right)^2 - \frac{1}{2} \left(\frac{a}{b\lambda} \right)^2 = \left(\frac{a}{R} \right)^2 \left(\frac{1}{8} \right)^2 \left[- \left(\frac{a}{b} \right)^2 \sin \frac{\pi x}{a} \left(1 + \cos \frac{\pi x}{b} \right) + \left(\frac{1}{8} \right)^2 \frac{1}{f} \left(1 - \cos \frac{2\pi x}{a} \right) \left(3 + 4 \cos \frac{\pi x}{b} + \cos \frac{2\pi x}{b} \right) \right] \\
&= \left(\frac{a}{R} \right)^2 \left(\frac{1}{8} \right)^2 \left[- \left(\frac{a}{b} \right)^2 \sin \frac{\pi x}{a} - \left(\frac{a}{b} \right)^2 \sin \frac{\pi x}{a} \cos \frac{\pi x}{b} + \frac{1}{64} \frac{\pi^2}{4} \left\{ 3 + 4 \cos \frac{\pi x}{b} + \cos \frac{2\pi x}{b} - 3 \cos \frac{2\pi x}{a} - 4 \cos \frac{2\pi x}{a} \cos \frac{\pi x}{b} \right. \right. \\
&\quad \left. \left. - \cos \frac{2\pi x}{a} \cos \frac{2\pi x}{b} \right\} \right]
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{a}{R} \right)^2 \left(\frac{1}{8} \right)^2 \left[\left\{ - \left(\frac{a}{b} \right)^2 \sin \frac{\pi x}{a} + \frac{3}{64} \frac{\pi^2}{4} - \frac{3}{64} \frac{\pi^2}{4} \cos \frac{2\pi x}{a} \right\} \right. \\
&\quad \left. + \left\{ - \left(\frac{a}{b} \right)^2 \sin \frac{\pi x}{a} + \frac{1}{16} \frac{\pi^2}{4} - \frac{1}{16} \frac{\pi^2}{4} \cos \frac{2\pi x}{a} \right\} \cos \frac{\pi x}{b} \right. \\
&\quad \left. + \left\{ \frac{1}{64} \frac{\pi^2}{4} - \frac{1}{64} \frac{\pi^2}{4} \cos \frac{2\pi x}{a} \right\} \cos \frac{2\pi x}{b} \right]
\end{aligned}$$

$$\begin{aligned}
\frac{1}{gE^2 a b} \int_0^a \int_0^b \sigma_x^2 dy dx &= \left(\frac{R}{g}\right)^4 \left(\frac{H}{g}\right)^2 \left[\frac{3}{4} \left(\frac{3}{64}\right)^2 (H)^2 + \frac{1}{a} \frac{3}{64} H^3 \right] \left(\sin^2 \frac{\pi x}{a} - 1 \right) \left(\frac{\pi}{a} \right) \sin \frac{\pi x}{a} dx + \frac{3}{4} \int_0^a \left(\frac{\pi}{a} \right) \sin^2 \frac{\pi x}{a} dx \\
&+ \frac{3}{8} \left(\frac{H}{g} \right)^2 - \frac{1}{2a} \left(\frac{H}{g} \right) \int_0^a \left(1 - \cos \frac{2\pi x}{a} \right) \left(\frac{\pi}{a} \right) \sin \frac{\pi x}{a} dx + \frac{1}{8a} \left(\frac{H}{g} \right)^2 \int_0^a \left(\frac{\pi}{a} \right)^2 dx \\
&= \left(\frac{R}{g}\right)^4 \left(\frac{H}{g}\right)^2 \left[\frac{25}{16384} H^2 + \frac{3}{64} H \right] \left\{ -\frac{1}{3} - 1 \right\} + \frac{3}{4} \left\{ \frac{\pi^2}{6} - \frac{1}{4} \right\} - \frac{1}{32} \left\{ 1 + \frac{1}{3} \right\} + \frac{3}{16384} \left(\frac{H}{g} \right)^2 \int_0^a \left(\frac{\pi}{a} \right)^2 dx \\
&= \left(\frac{R}{g}\right)^4 \left(\frac{H}{g}\right)^2 \left[\frac{105}{32768} H^2 - \frac{5}{48} H \right] + \frac{3}{4} \left(\frac{\pi^2}{12} - \frac{1}{8} \right) \int_0^a \left(\frac{\pi}{a} \right)^2 dx
\end{aligned}$$

$$\frac{1}{2} \frac{\partial \omega}{\partial x} \frac{\partial \omega}{\partial y} = \left(\frac{R}{g}\right)^2 \left(\frac{H}{g}\right) \lambda \left[-\frac{1}{2} \left(\frac{\pi}{a}\right) \left(1 + \cos \frac{\pi x}{a}\right) \sin \frac{\pi y}{b} + \left(\frac{H}{16}\right) \left(\sin \frac{\pi x}{a} + \frac{1}{2} \sin \frac{2\pi x}{a} \right) \left(\sin \frac{\pi y}{b} + \frac{1}{2} \sin \frac{2\pi y}{b} \right) \right]$$

$$\frac{I_{xy}}{H} = \left(\frac{R}{g}\right)^2 \left(\frac{H}{g}\right) \lambda \left[-\frac{1}{2} \left(\frac{\pi}{a}\right) \sin \frac{\pi y}{b} - \frac{1}{2} \left(\frac{\pi}{a}\right) \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} + \left(\frac{H}{16}\right) \left\{ \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} + \frac{1}{2} \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} \right. \right.$$

$$\left. + \frac{1}{2} \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} + \frac{1}{4} \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{b} \right\} + \frac{1}{2} \sqrt{2} \lambda a_1 \sinh \frac{\sqrt{2} \lambda \pi x}{a} \sin \frac{\pi y}{b}$$

$$+ \frac{1 + \lambda^2}{(1 + 2\lambda^2)^2} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} + \frac{1}{2(1 + 2\lambda^2)} \left(\frac{\pi x}{a} \right) \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} + \frac{1}{2} \sqrt{2} \lambda a_1 \sinh \frac{\sqrt{2} \lambda \pi x}{a} \sin \frac{\pi y}{b}$$

$$+ \frac{1 + \lambda^2}{64} \left\{ -\frac{4}{(1 + 2\lambda^2)} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} - \frac{4}{(1 + 2\lambda^2)} \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} - \frac{2(1 + 2\lambda^2)}{(1 + \lambda^2)^2} \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} \right\}$$

$$\begin{aligned}
\frac{U_4}{E} &= \left(\frac{a^2}{R}\right) \left(\frac{1+b}{8}\right) \left[\left\{ \frac{1}{2}(a) \left(\frac{1+b}{8}\right) \cos \frac{\pi x}{a} + \frac{a^2}{16} \sin \frac{\pi x}{a} + \frac{a^2}{32} \sin \frac{2\pi x}{a} + \frac{1+b^2}{(1+b)^2} \sin \frac{\pi x}{a} + \frac{1+b^2}{2(1+b)^2} \left(\frac{\pi x}{a}\right) \sin \frac{\pi x}{a} \right\} \right. \\
&\quad + \frac{1}{2} \sqrt{2} \lambda a, \sinh \frac{\sqrt{2} \lambda \pi x}{a} - \frac{a^2}{16} \frac{1}{(1+b)^2} \sin \frac{\pi x}{a} - \frac{a^2}{16} \frac{1}{(2+b)^2} \sin \frac{2\pi x}{a} \left. \sin \frac{\pi x}{8} \right] \\
&\quad + \left\{ \frac{a^2}{32} \sin \frac{\pi x}{a} + \frac{a^2}{64} \sin \frac{2\pi x}{a} - \frac{a^2}{32} \frac{(1+b)^2}{(1+b)^2} \sin \frac{\pi x}{a} - \frac{a^2}{128} \frac{2+b^2}{1+b^2} \sin \frac{2\pi x}{a} \left. \sin \frac{2\pi x}{8} \right\} \right] \\
&= \left(\frac{a^2}{R}\right) \left(\frac{1+b}{8}\right) \left[\left\{ -\frac{1}{2}(a) \left(\frac{1+b}{8}\right) \cos \frac{\pi x}{a} + \frac{a^2}{8} \frac{1}{(1+b)^2} \sin \frac{\pi x}{a} + \frac{a^2}{32} \frac{1^2}{(2+b)^2} \sin \frac{2\pi x}{a} + \frac{1+b^2}{(1+b)^2} \sin \frac{\pi x}{a} + \frac{1+b^2}{2(1+b)^2} \sin \frac{2\pi x}{a} \right\} \right. \\
&\quad + \frac{1}{2} \sqrt{2} \lambda a, \sinh \frac{\sqrt{2} \lambda \pi x}{a} \left. \sin \frac{\pi x}{8} \right] \\
&\quad + \left\{ \frac{a^2}{16} \frac{3b^2}{(1+b)^2} \sin \frac{\pi x}{a} + \frac{a^2}{128} \frac{3b^2}{(1+b)^2} \sin \frac{2\pi x}{a} \left. \sin \frac{2\pi x}{8} \right\} \right] \\
&= \left(\frac{a^2}{R}\right) \left(\frac{1+b}{8}\right) \left[\left\{ \left[\frac{a^2}{8} \frac{1+b^2}{(1+b)^2} + \frac{1+b^2}{(1+b)^2} \right] \sin \frac{\pi x}{a} + \frac{a^2}{32} \frac{1^2}{(2+b)^2} \sin \frac{2\pi x}{a} + \frac{1}{2} \sqrt{2} \lambda a, \sinh \frac{\sqrt{2} \lambda \pi x}{a} \right. \right. \\
&\quad \left. \left. - \frac{1}{2}(a) \left(\frac{1+b}{8}\right) \cos \frac{\pi x}{a} - \frac{a^2}{(1+b)^2} \sin \frac{\pi x}{a} \right\} \sin \frac{\pi x}{8} \right. \\
&\quad \left. + \left\{ \frac{a^2}{16} \frac{3b^2}{(1+b)^2} \sin \frac{\pi x}{a} + \frac{a^2}{128} \frac{3b^2}{(1+b)^2} \sin \frac{2\pi x}{a} \left. \sin \frac{2\pi x}{8} \right\} \right] \right]
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{E'ab} \int_0^a \int_0^b xy^2 dx dy &= \left(\frac{a}{b}\right)^2 \left(\frac{b}{8}\right)^2 \left[\frac{1}{4} \left\{ \frac{1}{8} \frac{b^2}{(1+\lambda^2)^2} + \frac{1}{4} \frac{b^2}{(1+\lambda^2)^2} \right\}^2 + \frac{1}{4} \left(\frac{b^2}{32} \right)^2 \frac{b^2}{(2+\lambda^2)^2} \right] \\
&+ \frac{1}{4} \left(\frac{b^2}{16} \right)^2 \left(\frac{3b^2}{1+\lambda^2} \right)^2 + \frac{1}{4} \left(\frac{b^2}{128} \right)^2 \left(\frac{3\lambda^2}{1+\lambda^2} \right)^2 + \frac{1}{4} \left(\frac{b^2}{128} \right)^2 \left(\frac{3\lambda^2}{1+\lambda^2} \right)^2 \\
&+ \left[\frac{1}{8} \frac{b^2}{(1+\lambda^2)^2} + \frac{1}{(1+\lambda^2)^2} \right] \frac{1}{a} \int_0^a \sin \frac{\pi x}{a} \left\{ \frac{1}{2} \sqrt{2} \lambda a \sinh \frac{\sqrt{2} \lambda \pi x}{a} - \frac{1}{2} \frac{\pi x}{a} - \frac{b^2}{(1+\lambda^2)^2} \left(\frac{\pi x}{a} \right) \cos \frac{\pi x}{a} \right\} dx \\
&+ \frac{1}{32} \frac{b^2}{(2+\lambda^2)^2} \frac{1}{a} \int_0^a \sin \frac{2\pi x}{a} \left\{ \frac{1}{2} \sqrt{2} \lambda a \sinh \frac{\sqrt{2} \lambda \pi x}{a} - \frac{1}{2} \frac{\pi x}{a} - \frac{b^2}{(1+\lambda^2)^2} \left(\frac{\pi x}{a} \right) \cos \frac{\pi x}{a} \right\} dx \\
&+ \frac{1}{2} \frac{1}{a} \int_0^a \left\{ \frac{1}{2} \sqrt{2} \lambda a \sinh \frac{\sqrt{2} \lambda \pi x}{a} - \frac{1}{2} \frac{\pi x}{a} - \frac{b^2}{(1+\lambda^2)^2} \left(\frac{\pi x}{a} \right) \cos \frac{\pi x}{a} \right\}^2 dx
\end{aligned}$$

$$\begin{aligned}
\frac{1}{a} \int_0^a \sin \frac{\pi x}{a} \sinh(\sqrt{2} \lambda) \left(\frac{\pi x}{a} \right) dx &= \frac{1}{2\pi i} \int_0^a \left[\cosh(\sqrt{2} \lambda + i) - \cosh(\sqrt{2} \lambda - i) \right] dx \\
&= \frac{1}{2\pi i} \left[\frac{1}{\sqrt{2} \lambda + i} \sinh \theta (\sqrt{2} \lambda + i) - \frac{1}{\sqrt{2} \lambda - i} \sinh \theta (\sqrt{2} \lambda - i) \right]_0^a = + \frac{1}{\pi} \frac{1}{(1+\lambda^2)^2} \sinh(\sqrt{2} \lambda \pi)
\end{aligned}$$

$$\frac{1}{a} \int_0^a \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a} dx = \frac{1}{\pi} \left[\sin \theta - \theta \cos \theta \right]_0^a = 1$$

$$\frac{1}{a} \int_0^a \left(\frac{\pi x}{a} \right) \cos \frac{\pi x}{a} dx = \frac{1}{\pi} \left[\sin \theta - \theta \cos \theta \right]_0^a = -\frac{1}{4}$$

$$\frac{\Delta \rho}{\rho b t E} = -\left(\frac{\rho}{R}\right)^2 \frac{(H_0)^2}{64} \left(\frac{3}{2} \pi^2 \frac{\sigma}{E}\right)$$

$$\frac{t^2}{12ab} \int_0^a \int_0^b \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w_0}{\partial x^2} \right)^2 dx dy = \frac{1}{12} \left(\frac{t}{R} \right)^2 \left(\frac{H_0^2}{2} \right)^2 \frac{3}{4}$$

$$\frac{t^2}{12ab} \int_0^a \int_0^b \left(\frac{\partial^2 w}{\partial y^2} \right)^2 dx dy = \frac{1}{12} \left(\frac{t}{R} \right)^2 \left(\frac{H_0^2}{2} \right)^2 \frac{3}{4}$$

$$\frac{t^2}{12ab} \int_0^a \int_0^b \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 dx dy = \frac{1}{12} \left(\frac{t}{R} \right)^2 \left(\frac{H_0^2}{2} \right)^2 \frac{1}{4}$$

The part of extensional energy which is independent of $\alpha_3 b_5$

$$= \left(\frac{\rho}{R}\right) \frac{(f\pi)^2}{64} \left[(-\pi)^2 \left\{ \frac{1}{32} + \frac{1}{16(1+\lambda^2)^4} + \frac{1}{4(4+\lambda^2)^4} + \frac{1}{64(1+\lambda^2)^4} + \frac{1}{512} + \frac{1}{32\lambda^4} + \frac{1}{16(1+\lambda^2)^4} + \frac{\lambda^4}{512\lambda^4} + \right. \right.$$

$$\left. + \frac{\lambda^4 \nu}{64(4+\lambda^2)^4} + \frac{\lambda^4 \nu}{4(1+\lambda^2)^4} + \frac{2\lambda^2 \nu}{16(1+\lambda^2)^4} + \frac{2\lambda^2 \nu}{16(4+\lambda^2)^4} + \frac{2\lambda^2 \nu}{16(1+4\lambda^2)^4} \right\}$$

$$- (f\pi)^2 \left\{ \frac{1}{4(1+\lambda^2)^4} + \frac{1}{4\lambda^4} + \frac{\lambda^4}{4(1+\lambda^2)^4} + \frac{2\lambda^4}{4(1+\lambda^2)^4} \right\}$$

$$+ \left\{ \frac{1}{4(1+\lambda^2)^4} + \frac{1}{2\lambda^4} + \frac{\lambda^4}{4(1+\lambda^2)^4} + \frac{2\lambda^2}{4(1+\lambda^2)^4} \right\}$$

$$= \left(\frac{\rho}{R}\right) \frac{(f\pi)^2}{64} \left[(f\pi)^2 \left\{ \frac{17}{512} + \frac{17}{512\lambda^4} + \frac{1}{16(1+\lambda^2)^2} + \frac{1}{64(4+\lambda^2)^2} + \frac{1}{64(1+4\lambda^2)^2} \right\} \right.$$

$$\left. - (f\pi)^2 \left\{ \frac{1}{4\lambda^4} + \frac{1}{4(1+\lambda^2)^2} \right\} + \left\{ \frac{1}{2\lambda^4} + \frac{1}{4(1+\lambda^2)^2} \right\} \right]$$

The coefficient of $\left\{ a_2 \frac{(f \lambda^2)^2}{64} \frac{\sin k 2\pi}{2\pi} \right\} \left(\frac{e}{R} \right)^4$ in $(A_1 + A_2 + 2A_3)$

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$$(f \lambda^2) \left\{ -\frac{1}{4} - \frac{1}{4(1+\lambda^2)^2} - \frac{1}{20(1+4\lambda^2)^2} + \frac{1}{4} + \frac{\lambda^4}{4(1+\lambda^2)^2} + \frac{\lambda^4}{5(1+4\lambda^2)^2} \right. \\ \left. - \frac{2\lambda^2}{4(1+\lambda^2)^2} - \frac{2\lambda^2}{5(1+4\lambda^2)^2} \right\}$$

$$= -\frac{1}{4(1+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2} - \frac{1}{20(1+4\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)}$$

$$+ \frac{1}{2(1+\lambda^2)^2} - 1 - \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{2\lambda^2}{2(1+\lambda^2)^2}$$

$$= \left\{ -1 + \frac{1}{2(1+\lambda^2)^2} + \frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2} \right\}$$

$$\left\{ b_2 \frac{(f \lambda^2)^2}{64} \frac{\sin k 2\pi}{2\pi} \right\} \left(\frac{e}{R} \right)^4$$

$$\left\{ \frac{1}{2} + \frac{1}{4} - \frac{3}{100(1+4\lambda^2)^2} + \frac{1}{4} + \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{2}{5} \frac{\lambda^4}{(1+4\lambda^2)^2} + \frac{3}{25} \frac{\lambda^4}{(1+4\lambda^2)^2} \right. \\ \left. - \frac{2\lambda^2}{5(1+4\lambda^2)^2} + \frac{4\lambda^2}{25(1+4\lambda^2)^2} \right\}$$

$$= (f \lambda^2) \left\{ \frac{1}{2} + \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{2}{5} \frac{\lambda^2(\lambda^2-1)}{(1+4\lambda^2)^2} + \frac{1}{100} \frac{(2\lambda^2+3)(6\lambda^2-1)}{(1+4\lambda^2)^2} \right\}$$

$$\left\{ -1 - \frac{\lambda^4}{(1+\lambda^2)^2} \right\} =$$

$$\begin{aligned}
 & \left\{ b_2 \frac{(4\lambda^2)^2}{64} \cosh 2\pi \right\} \left(\frac{0}{16} \right)^4 \\
 & (4\pi^2) \left\{ -\frac{1}{4} - \frac{1}{4(1+\lambda^2)^2} - \frac{1}{20(1+4\lambda^2)^2} + \frac{1}{4} + \frac{\lambda^4}{4(1+\lambda^2)} + \frac{\lambda^4}{5(1+4\lambda^2)^2} \right. \\
 & \quad \left. - \frac{2\lambda^2}{4(1+\lambda^2)^2} - \frac{2\lambda^2}{5(1+4\lambda^2)^2} \right\} \\
 & = (4\pi^2) \left\{ -\frac{1}{4(1+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2} - \frac{1}{20(1+4\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)^2} \right\} \\
 & \hline
 & \left\{ \frac{1}{2(1+\lambda^2)^2} - 1 - \frac{\lambda^4}{2(1+\lambda^2)} + \frac{2\lambda^2}{2(1+\lambda^2)^2} \right\} \\
 & = \frac{1}{2(1+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2} - 1
 \end{aligned}$$

$$\begin{aligned}
 & \left\{ a_2 \frac{(4\lambda^2)^2}{64} \frac{\sinh 4\pi}{4\pi} \right\} \left(\frac{0}{16} \right)^4 \\
 & (4\pi^2) \left\{ -\frac{1}{16} - \frac{1}{5} \frac{1}{(4+\lambda^2)^2} + \frac{1}{16} + \frac{\lambda^4}{5(4+\lambda^2)^2} - \frac{2\lambda^2}{5(4+\lambda^2)^2} \right\} \\
 & = (4\pi^2) \left\{ -\frac{1}{5} \frac{1}{(4+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2} \right\}
 \end{aligned}$$

$$\begin{aligned}
 & \left\{ b_4 \frac{(4\lambda^2)^2}{64} \frac{\sinh 4\pi}{4\pi} \right\} \left(\frac{0}{16} \right)^4 \\
 & (4\pi^2) \left\{ +\frac{1}{16} + \frac{12}{25} \frac{1}{(4+\lambda^2)^2} + \frac{1}{16} + \frac{2\lambda^4}{5(4+\lambda^2)^2} - \frac{3\lambda^2}{25(4+\lambda^2)^2} - \frac{2\lambda^2}{5(4+\lambda^2)^2} \right. \\
 & \quad \left. + \frac{46\lambda^2}{25(4+\lambda^2)^2} \right\} \\
 & = (4\pi^2) \left\{ \frac{1}{8} + \frac{(2+3\lambda^2)(6-\lambda^2)}{25(4+\lambda^2)^2} - \frac{2}{5} \frac{\lambda^2(1-\lambda^2)}{(4+\lambda^2)^2} \right\}
 \end{aligned}$$

$$\left\{ b_4 \frac{4\lambda^2}{64} \cosh 2\pi \right\} \left(\frac{a^4}{8} \right)$$

$$H\pi \left\{ -\frac{1}{16} - \frac{4}{5} \frac{1}{(4+\lambda^2)^2} + \frac{1}{16} + \frac{1}{5} \frac{\lambda^4}{(4+\lambda^2)^2} - \frac{2\lambda^2}{5(4+\lambda^2)^2} \right\}$$

$$= H\pi \left\{ -\frac{1}{5} \frac{1}{(4+\lambda^2)} - \frac{1}{5} \frac{\lambda^2(1-\lambda^2)}{(4+\lambda^2)^2} \right\}$$

Terms in $A_1 + A_2 + 2A_3$ involving quadratics of a 's & b 's. $\div \frac{4\lambda^2(1-\lambda^2)}{64}$

$$\frac{1}{4} \left\{ (1+\lambda^2)^2 \frac{\sinh 4\pi}{4\pi} + (1-\lambda^2)^2 \right\} a_2^2$$

$$+ \frac{1}{4} \left\{ (3\lambda^4 + 2\lambda^2 - 1) \frac{\sinh 4\pi}{4\pi} + (\lambda^2 + 1)^2 \cosh 2\pi - 4(1-\lambda^2)\lambda \right\} a_2 b_2$$

$$+ \frac{1}{4} \left\{ [4\pi^2(\lambda^2 + 1)^2 + \frac{1}{2}(5\lambda^4 + 2\lambda^2 + 1)] \frac{\sinh 4\pi}{4\pi} + \frac{1}{2}(3\lambda^4 + 2\lambda^2 - 1) \cosh 4\pi \right. \\ \left. - \frac{4\pi^2}{3}(1-\lambda^2)^2 + 2(2\lambda^4 - 1) \right\} b_2^2$$

$$\frac{1}{4} \left\{ (1+\lambda^2)^2 \frac{\sinh 8\pi}{8\pi} + (1-\lambda^2)^2 \right\} a_4^2$$

$$+ \frac{1}{4} \left\{ (3\lambda^4 + 2\lambda^2 - 1) \frac{\sinh 8\pi}{8\pi} + (\lambda^2 + 1)^2 \cosh 4\pi + 4\lambda^2(\lambda^2 - 1) \right\} a_4 b_4$$

$$+ \frac{1}{4} \left\{ [16\pi^2(\lambda^2 + 1)^2 + \frac{1}{2}(5\lambda^4 + 2\lambda^2 + 1)] \frac{\sinh 8\pi}{8\pi} + \frac{1}{2}(3\lambda^4 + 2\lambda^2 - 1) \cosh 8\pi \right. \\ \left. - \frac{16\pi^2}{3}(1-\lambda^2)^2 + 2(2\lambda^4 - 1) \right\} b_4^2$$

①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩ ²
λ	$\sqrt{2}\lambda/\pi$	$\lambda^2 \div 2.302585$	e^{λ^2}	$e^{-\lambda^2}$	$e^{\lambda^2/\pi}$	$\sin(\lambda^2/\pi)$	$\sin(\lambda^2/\pi)/\sqrt{2\pi}$	②/⑧	
2.0	8.885768	3.8590400	7228.363	0	3614.182	3614.182	406.7353	8.885768	
1.9	8.441480	3.6660882	4635.411	0	2317.706	2317.706	274.5616	8.441480	
1.8	7.992191	3.4731359	2720.515	0	1486.298	1486.298	185.8525	7.992191	
1.7	7.552903	3.2801841	1706.268	0.001	953.1345	953.1335	126.1943	7.552912	
1.6	7.108614	3.0872318	1222.452	0.001	611.2285	611.2255	85.98378	7.108626	
1.5	6.664326	2.8942800	785.9349	0.0013	391.9681	391.9668	58.81468	6.664460	
1.4	6.220038	2.7013242	502.7223	0.0020	251.3622	251.3602	40.41136	6.220088	
1.3	5.775749	2.5083759	322.3858	0.0031	161.1945	161.1914	27.90831	5.775860	
1.2	5.331461	2.3154241	206.7398	0.0049	103.3724	103.3675	19.38821	5.331714	
1.1	4.887172	2.1224718	132.5781	0.0076	66.29285	66.28525	13.56311	4.887732	
1.0	4.442884	1.9295200	85.01977	0.01176	42.51577	42.50401	9.566761	4.444113	
0.9	3.998596	1.7365662	54.52155	0.01814	27.26995	27.25161	6.815295	4.001287	
0.8	3.554307	1.5436159	34.76358	0.02860	17.49169	17.46249	4.914457	3.560127	
0.7	3.110019	1.3506641	22.72147	0.04460	11.23304	11.18844	3.597547	3.122416	
0.6	2.665730	1.1577118	14.25874	0.06955	7.223995	7.154645	2.683860	2.691644	
0.5	2.221442	0.9647600	9.210617	0.108453	4.664536	4.556053	2.050857	2.274322	
0.4	1.777154	0.7718082	5.713004	0.169119	3.044062	2.871743	1.616035	1.881005	

	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
2	2^4	2^2	1472	1472	1472	130×10^2	$(1+2 \times 10^2)^2$	$642 \times 10^4 \times 10^4$	$130 \times 10^2 / 655 \times 10^4$	$32 \times 10^2 / 64 \times 10^4$
2.0	16.0000	4.00	0.01519336	9.00	33.00	134.00	81.00	0.0000133164	0.000221864	0.0000361670
1.9	13.0321	3.61	0.01689114	8.72	29.88	133.61	67.584	0.0000134244	0.0002450203	0.0000391314
1.8	10.4976	3.24	0.01521716	7.78	24.92	133.24	55.9504	0.0000135140	0.0002718023	0.0000424134
1.7	8.3521	2.89	0.0121208	6.28	24.12	132.89	45.9164	0.0000135832	0.0002990768	0.0000460448
1.6	6.5536	2.56	0.0095000	4.12	21.48	132.56	37.4544	0.0000137192	0.0003305074	0.0000500609
1.5	5.0625	2.25	0.00733152	5.50	19.00	132.25	30.2500	0.0000143446	0.0003619045	0.0000544277
1.4	3.8416	1.96	0.0055873	4.92	16.18	131.96	24.0640	0.0000144254	0.000402581	0.000059654
1.3	2.8561	1.69	0.0044016	4.38	14.52	131.69	19.1644	0.0000142915	0.0004543760	0.000064508
1.2	2.0736	1.44	0.00300586	3.88	12.52	131.44	15.0544	0.00001513204	0.0005119111	0.0000700514
1.1	1.4641	1.21	0.00212234	3.41	10.18	131.21	11.6764	0.00001563597	0.0005854110	0.0000757115
1.0	1.0000	1.00	0.00144929	3.00	9.00	131.00	9.0000	0.00001627604	0.0006663005	0.0000813802
0.9	0.8161	0.81	0.00115107	2.72	7.48	130.81	6.8644	0.0000171090	0.0007618329	0.0000864519
0.8	0.6496	0.64	0.00089325	2.28	6.12	130.64	5.1984	0.0000175552	0.0008743018	0.000091920
0.7	0.4901	0.49	0.00063485	1.98	4.92	130.49	3.9204	0.0000185034	0.0010056158	0.0000975434
0.6	0.3296	0.36	0.00048787	1.72	3.88	130.36	2.9584	0.0000216456	0.0011564743	0.000102165
0.5	0.2525	0.25	0.00030599	1.50	3.00	130.25	2.2500	0.0000246146	0.0013248715	0.0001081382
0.4	0.1656	0.16	0.0002109	1.36	2.28	130.16	1.8496	0.0000291555	0.001460359	0.000115352

λ	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)
	$\lambda^2(1+\lambda^2)$	H_1	$3+2\lambda^2$	$11\lambda^2$	$13+2\lambda^2(1+\lambda^2)$	$(1+2\lambda^2)^2$	$(4+\lambda^2)$	$(2+\lambda^2)^2$	$5+\lambda^2$
9.0	0.00442355	0.03082446	11.00	22.66	235.0000	739.0000	8.00	36.0000	13.00
1.9	0.003917245	0.02010273	10.22	22.66	231.8852	555.4422	7.61	31.4724	12.22
1.8	0.003440936	0.02186244	9.48	20.37	193.7412	448.5080	7.24	27.4876	11.48
1.7	0.002974142	0.01830857	8.78	15.34	161.0252	311.6658	6.89	23.9124	10.78
1.6	0.002852145	0.01558961	8.12	12.46	132.8432	249.2209	6.56	20.7736	10.12
1.5	0.002204003	0.01274767	7.50	14.50	108.7800	166.3750	6.25	18.0625	9.50
1.4	0.00185207	0.01062978	6.92	12.11	88.2492	118.3749	5.96	15.6716	8.92
1.3	0.001536650	0.00888116	6.38	11.14	71.0232	84.9267	5.69	13.8641	8.38
1.2	0.001244518	0.00745872	5.88	9.74	56.6832	58.4107	5.44	11.8336	7.88
1.1	0.000990927	0.00631761	5.38	8.14	44.7672	40.0019	5.24	10.3041	7.42
1.0	0.000763957	0.00541789	5.00	7.00	35.0000	27.0000	5.00	9.0000	7.00
0.9	0.00057764	0.00472318	4.62	6.15	27.0232	17.9473	4.81	7.8961	6.62
0.8	0.000402401	0.00420050	4.28	5.51	20.7152	11.65235	4.64	6.9196	6.28
0.7	0.00028146	0.00382054	3.98	4.91	15.6812	7.762392	4.49	6.2001	5.91
0.6	0.000164235	0.00355845	3.70	4.46	11.7552	5.086498	4.36	5.5696	5.72
0.5	0.000094423	0.00338437	3.42	4.00	7.73500	3.37500	4.25	5.0625	5.50
0.4	0.000037753	0.00328121	3.16	3.61	6.5856	2.515456	4.16	4.6556	5.36

	(23)	(28)	(24)	(25)	(26)	(27)	(28)	(29)
λ	$-\frac{3}{4}(1+\lambda^2)$	$-\frac{3}{4}(1+\lambda^2)$	$\lambda^4(6\lambda+63)$	H_2	$3+2\lambda$	$(1+2\lambda)^3$	$4+11\lambda+10\lambda^2$	$1+3\lambda^2$
2.0	-0.0520633	-0.00047361	-0.0900110	-0.2410528	19.0000	72.9000	248.0000	13.00
1.9	-0.0070555	-0.00046201	-0.08036578	-0.2266372	16.0321	555.4122	194.0310	11.63
1.7	-0.0022211	-0.00046918	-0.07100250	-0.2131360	13.4976	418.5090	144.6160	10.72
1.8	-0.0019377	-0.00053259	-0.06219196	-0.2002259	11.3521	311.6658	119.3110	9.67
1.6	-0.0076514	-0.00051009	-0.05393877	-0.1881055	9.5536	229.2209	97.6960	8.68
1.5	-0.0822227	-0.000612745	-0.04624833	-0.1767822	8.0625	166.3250	79.3750	7.75
1.4	-0.0952439	-0.00057818	-0.03912491	-0.1662674	6.8416	118.3949	63.9760	6.88
1.3	-0.01070255	-0.000705736	-0.03258179	-0.1565532	5.8561	84.0277	51.1510	6.7
1.2	-0.0120116	-0.00057025	-0.02662131	-0.1471130	5.0736	58.4107	40.5760	5.32
1.1	-0.013366140	-0.00061127	-0.02125494	-0.1376013	4.4641	40.0069	31.9510	4.63
1.0	-0.01535000	-0.00068056	-0.01649306	-0.1323277	4.0000	27.0000	25.0000	4.00
0.9	-0.017891221	-0.000926750	-0.01233447	-0.1260054	3.6561	17.98473	19.4710	3.43
0.8	-0.020559211	-0.00098627	-0.0082509	-0.1204918	3.4596	11.85235	15.1360	2.92
0.7	-0.023674262	-0.00104580	-0.00593529	-0.1154442	3.2401	7.762392	11.7710	2.47
0.6	-0.0255907	-0.00103461	-0.00367499	-0.110605	3.1296	5.086448	9.2560	2.01
0.5	-0.03125000	-0.001157408	-0.00202546	-0.1071219	3.0625	3.38000	7.3750	1.25
0.4	-0.03446972	-0.001205433	-0.00091322	-0.1047547	3.0256	2.55456	6.0160	1.48

H₃

	(30)	(31)	(32)	(33)	(34)	(35)
λ	$\frac{3+2\lambda}{1+2\lambda}$	$-\frac{3(4+11\lambda+4\lambda^2)}{1+2\lambda}$	$\frac{I^2-1732\lambda}{24-1023\lambda}$	$\lambda^2(6\lambda+10)$	$-\frac{9}{16(1+2\lambda)^2}$	$\frac{I^2}{2} + (63) + (30)$
2.0	+0.00515735	-0.469481461	+0.594003923	+2.472153065	-0.00785630	+1.7019974
1.9	+0.00216307	-0.442930075	+0.579183674	+0.41901532	-0.00786627	+1.6450938
1.8	+0.00619109	-0.44634605	+0.589361427	+0.315438667	-0.00793335	+1.5902059
1.7	+0.009105918	-0.446156323	+0.588523374	+0.314931938	-0.0102616	+1.5383633
1.6	+0.0041914	-0.448907415	+0.583252210	+0.27222546	-0.0116213	+1.4496100
1.5	+0.01211495	-0.49199360	+0.579465415	+0.27061927	-0.01321955	+1.4440008
1.4	+0.0144467	-0.498443211	+0.575058226	+0.178402561	-0.01615507	+1.3959471
1.3	+0.0174233	-0.49992626	+0.5679905814	+0.147701113	-0.0188192	+1.3625918
1.2	+0.02141506	-0.505312202	+0.563856270	+0.115493944	-0.02213320	+1.3270593
1.1	+0.02749945	-0.51219256	+0.556328420	+0.082746369	-0.02611722	+1.2952293
1.0	+0.03703706	-0.520833333	+0.548311360	+0.064515067	-0.03086190	+1.2673538
0.9	+0.0502228	-0.53164225	+0.538370601	+0.04149733	-0.0343151	+1.2437190
0.8	+0.07191823	-0.54593211	+0.526617491	+0.036795046	-0.0420316	+1.2245924
0.7	+0.10465550	-0.563925237	+0.513003432	+0.016161041	-0.05002914	+1.2091237
0.6	+0.15376005	-0.586134667	+0.497305852	+0.02319573	-0.05666444	+1.2003114
0.5	+0.2246515	-0.614583333	+0.47972244	+0.005010139	-0.06317561	+1.1935552
0.4	+0.30070095	-0.60981592	+0.447514831	+0.002137310	-0.06358824	+1.1922497

49-

λ	(66) $\frac{2}{\lambda}(\frac{1}{\lambda} + \frac{1}{\lambda^2})$	(67) $\frac{512}{\lambda} \cdot \frac{H}{\lambda^2}$	(68) $8H_2/\lambda^2$	(69) H_3/λ^2	(70) $\frac{67}{\lambda}$	(71) $\frac{30}{\lambda} \times 0.9$	(72) $k_0 = 2.5 \sqrt{Q}$
2.0	1.0925925	0.4383923	-0.4451046	0.4250993	0.185356	0.4648923	1.31367
1.9	1.0119277	0.4099217	-0.5021813	0.1557047	0.181815	0.4611402	1.35814
1.8	0.9367372	0.3838123	-0.5265117	0.1406043	0.1664037	0.4572537	1.35610
1.7	0.8450416	0.3603994	-0.5545565	0.1323056	0.1918426	0.4498204	1.34138
1.6	0.7638425	0.3392191	-0.5878217	0.121289	0.1977045	0.4473790	1.32782
1.5	0.7049135	0.3223115	-0.6298551	0.1117781	0.2061525	0.439527	1.31670
1.4	0.6920823	0.3085218	-0.6786222	0.1222144	0.2197377	0.4466448	1.40722
1.3	0.6551961	0.2989582	-0.7410602	0.12173	0.2410462	0.5282632	1.45364
1.2	0.6224691	0.2946655	-0.8203522	0.1215810	0.2715546	0.5732142	1.51478
1.1	0.606917	0.2970263	-0.9229838	1.2704374	0.3179421	0.6430027	1.62374
1.0	0.5925925	0.3021177	-1.0590288	1.267358	0.3906209	0.7510244	1.73324
0.9	0.6024165	0.331240	-1.2444728	1.5354556	0.5093475	0.9251066	1.92364
0.8	0.6325925	0.373378	-1.5061475	1.9134256	0.7164306	1.279258	2.20908
0.7	0.710517	0.4435138	-1.8913331	2.4191300	1.095267	1.754445	2.64910
0.6	0.845420	0.5620069	-2.4902335	3.3334206	1.8734054	2.818104	3.35748
0.5	1.0925925	0.7701322	-3.4919008	4.7741408	3.676796	5.211904	4.56778
0.4	1.525925	1.166524	-5.3477450	7.415506	8.6793811	11.712673	6.84638

496

$$K = 2 \left[A \left(\frac{f}{E} \right)^2 + B \right]^{\frac{1}{2}} - C \left(\frac{f}{E} \right)$$

497

$$2A \left(\frac{f}{E} \right) = C \left[A \left(\frac{f}{E} \right)^2 + B \right]^{\frac{1}{2}}$$

$$(4A^2 - C^2A) \left(\frac{f}{E} \right)^2 = C^2B$$

$$\left(\frac{f}{E} \right)^2 = \frac{C^2B}{4A^2 - C^2A}$$

$$\boxed{\left(\frac{f}{E} \right) = \sqrt{\frac{B}{\left(\frac{2A}{C} \right)^2 - A}}$$

$$K_{min} = \left(\frac{4A}{C} - C \right) \sqrt{\frac{C^2B}{4A^2 - C^2A}}$$

$$\boxed{K_{min} = \left[2 \left(\frac{2A}{C} \right) - C \right] \sqrt{\frac{B}{\left(\frac{2A}{C} \right)^2 - A}}$$

λ	(43)	(44)	(45)	(46)	(47)	(48)	(49)	(50)
	$2A/C$	$(43)^2 - A$	$(44) \div (44)$	$\frac{1}{1} - (45) \frac{1}{2}$	$2(47) - C$	Known		
2.0	0.7738271	0.4122663	1.127602					
1.9	0.7432249	0.365837	1.241341					
1.8	0.7160008	0.3242534	1.443844					
1.7	0.6922496	0.2873669	1.565317					
1.6	0.6726591	0.2547458	1.835957					
1.5	0.6581802	0.2263487	2.117762					
1.4	0.6476062	0.1996539	2.486637					
1.3	0.6405104	0.1821236	2.900525					
1.2	0.6360457	0.1667449	3.440171					
1.1	0.6348571	0.1567138	4.103039					
1.0	0.737695	0.1535752	4.891271	2.211376	0.920744			
0.9	0.6185591	0.1606915	5.757035					
0.8	0.9446661	0.1855747	6.574095					
0.7	1.1581421	0.2460764	7.129176					
0.6	1.5046023	0.3904227	7.216280					
0.5	2.1058557	0.7579066	6.882348					
0.4	3.2512325	1.8771317	1.242146					

Mistake made !!!

478

$$\frac{w}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left[1 - \left(\frac{y}{a} \right)^2 - \frac{1}{4} (1 + \cos \frac{2\pi x}{a}) (1 + \cos \frac{2\pi y}{a}) \right] \quad \underline{\underline{451}}$$

$$\frac{w_0}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left[1 - \left(\frac{y}{a} \right)^2 \right]$$

$$R \frac{\partial^2 w}{\partial x^2} = \left[-1 + \frac{1}{2} \pi^2 \cos \frac{2\pi x}{a} (1 + \cos \frac{2\pi y}{a}) \right]$$

$$R \frac{\partial^2 w}{\partial y^2} = \left[+ \frac{1}{2} \pi^2 (1 + \cos \frac{2\pi x}{a}) \cos \frac{2\pi y}{a} \right]$$

$$R \frac{\partial^2 w}{\partial x \partial y} = \left[- \frac{1}{2} \pi^2 \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{a} \right]$$

$$R \frac{\partial^2 w_0}{\partial x^2} = [-1], \quad R \frac{\partial^2 w_0}{\partial y^2} = R \frac{\partial^2 w_0}{\partial x \partial y} = 0$$

Thus

$$\nabla^4 F = \frac{E}{R^2} \left[\frac{1}{4} \pi^2 + \sin^2 \frac{2\pi x}{a} \sin^2 \frac{2\pi y}{a} \right.$$

$$\left. + \frac{1}{2} \pi^2 (1 + \cos \frac{2\pi x}{a}) \cos \frac{2\pi y}{a} - \frac{1}{4} \pi^4 (1 + \cos \frac{2\pi x}{a}) \cos \frac{2\pi y}{a} (1 + \cos \frac{2\pi y}{a}) \cos \frac{2\pi x}{a} \right]$$

$$= \frac{E}{R^2} \left[\frac{1}{2} \pi^2 (1 + \cos \frac{2\pi x}{a}) \cos \frac{2\pi y}{a} + \frac{1}{4} \pi^4 \left\{ \frac{1}{4} (1 - \cos \frac{4\pi x}{a}) (1 - \cos \frac{4\pi y}{a}) \right. \right. \\ \left. \left. - \frac{1}{4} (2 \cos \frac{2\pi x}{a} + 1 + \cos \frac{4\pi x}{a}) (2 \cos \frac{2\pi y}{a} + 1 + \cos \frac{4\pi y}{a}) \right\} \right]$$

$$\begin{aligned}
 \nabla^4 F &= \frac{E}{R^2} \left[\frac{f}{2} \pi^2 (1 + \cos \frac{2\pi x}{a}) \cos \frac{2\pi y}{a} + \frac{f^2}{16} \pi^4 \left\{ x - 2 \cos \frac{4\pi x}{a} - 2 \cos \frac{4\pi y}{a} + \cos \frac{4\pi x}{a} \cos \frac{4\pi y}{a} \right. \right. \\
 &\quad - 4 \cos \frac{2\pi x}{a} \cos \frac{2\pi y}{a} - 2 \cos \frac{4\pi x}{a} \cos \frac{2\pi y}{a} - 2 \cos \frac{2\pi x}{a} \cos \frac{4\pi y}{a} - \left. \left. \cos \frac{4\pi x}{a} \cos \frac{4\pi y}{a} \right\} \right] \\
 &= \frac{E}{R^2} \left[\frac{f}{2} \pi^2 (1 + \cos \frac{2\pi x}{a}) \cos \frac{2\pi y}{a} - \frac{f^2}{8} \pi^4 \left\{ \cos \frac{2\pi x}{a} + \cos \frac{2\pi y}{a} + 2 \cos \frac{2\pi x}{a} \cos \frac{2\pi y}{a} \right. \right. \\
 &\quad \left. \left. + \cos \frac{4\pi x}{a} + \cos \frac{4\pi y}{a} + \cos \frac{4\pi x}{a} \cos \frac{2\pi y}{a} + \cos \frac{2\pi x}{a} \cos \frac{4\pi y}{a} \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 \nabla^4 F &= \frac{E}{R^2} \left[-\frac{f\pi^2}{4} \cos \frac{2\pi x}{a} + \left\{ 1 - \frac{f\pi^2}{4} \right\} \cos \frac{2\pi y}{a} + \left\{ 1 - \frac{f\pi^2}{4} \right\} \cos \frac{2\pi x}{a} \cos \frac{2\pi y}{a} \right. \\
 &\quad \left. - \frac{f\pi^2}{4} \left\{ \cos \frac{4\pi x}{a} + \cos \frac{4\pi y}{a} + \cos \frac{4\pi x}{a} \cos \frac{2\pi y}{a} + \cos \frac{2\pi x}{a} \cos \frac{4\pi y}{a} \right\} \right]
 \end{aligned}$$

$$\nabla^4 F = \frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = C \cos \frac{2\pi x}{a}, \quad \text{Put } F = C \cos \frac{2\pi x}{a}$$

$$C \left\{ \left(\frac{2\pi}{a} \right)^4 - 2 \left(\frac{2\pi}{a} \right)^2 \right\} = C$$

hence the particular integral is

$$E(k)^2 \frac{1}{8} \left[-\frac{1}{4} \frac{1}{(2\pi)^2} \cos \frac{2\pi x}{a} + (1 - \frac{1}{4}) \frac{1}{(2\pi)^2} \cos \frac{2\pi x}{a} + \frac{1}{4} (1 - \frac{1}{4}) \frac{1}{(2\pi)^2} \cos \frac{2\pi x}{a} \right. \\ \left. - \frac{1}{4} \frac{1}{(4\pi)^2} \cos \frac{4\pi x}{a} + \frac{1}{4} \frac{1}{(4\pi)^2} \cos \frac{4\pi x}{a} + \frac{1}{4} \frac{1}{(4\pi)^2} \cos \frac{4\pi x}{a} + \frac{1}{25} \frac{1}{(2\pi)^2} \cos \frac{2\pi x}{a} \right]$$

The complementary function can be taken as

$$F = \frac{E}{(2\pi)^2} \left[A_2 \cosh \left(\frac{2\pi y}{a} \right) + B_2 \left(\frac{2\pi y}{a} \right) \sinh \left(\frac{2\pi y}{a} \right) \right] \cos \frac{2\pi x}{a} \\ + \frac{E}{(4\pi)^2} \left[A_4 \cosh \left(\frac{4\pi y}{a} \right) + B_4 \left(\frac{4\pi y}{a} \right) \sinh \left(\frac{4\pi y}{a} \right) \right] \cos \frac{4\pi x}{a} \\ + E a^2 \left(\frac{y}{a} \right)^2 C$$

$$\begin{aligned} \frac{\sigma_y}{E} = & - \left[A_2 \cosh\left(\frac{2\pi x}{a}\right) + B_2 \left(\frac{2\pi x}{a}\right) \sinh\left(\frac{2\pi x}{a}\right) \right] \cos\frac{2\pi y}{a} - \left[A_4 \cosh\left(\frac{4\pi x}{a}\right) + B_4 \left(\frac{4\pi x}{a}\right) \sinh\left(\frac{4\pi x}{a}\right) \right] \cos\frac{4\pi y}{a} \\ & + 2C + \left(\frac{a}{R}\right)^2 \frac{1}{8} \left[\frac{\frac{1}{4}\pi^2}{4} \cos\frac{2\pi x}{a} + \frac{1}{4} \left(\frac{\frac{1}{4}\pi^2}{2} - 1\right) \cos\frac{2\pi x}{a} \cos\frac{2\pi y}{a} \right. \\ & \left. + \frac{\frac{1}{4}\pi^2}{4} \cos\frac{4\pi x}{a} + \frac{1}{25} \cos\frac{4\pi x}{a} \cos\frac{2\pi y}{a} + \frac{1}{25} \cos\frac{2\pi x}{a} \cos\frac{4\pi y}{a} \right] \end{aligned}$$

The condition is $\frac{\sigma_y}{E} = -\frac{\sigma}{E}$, when $y = \pm a$.

$$\text{Thus} \quad -\frac{\sigma}{E} = 2C \quad \boxed{C = -\frac{\sigma}{2E}}$$

$A_2 \cosh 2\pi + 2\pi B_2 \sinh 2\pi = \left(\frac{a}{R}\right)^2 \frac{1}{8} \left[\frac{\frac{1}{4}\pi^2}{4} + \frac{1}{4} \left(\frac{\frac{1}{4}\pi^2}{2} - 1 \right) + \frac{\frac{1}{4}\pi^2}{4} \left(\frac{1}{25} \right) \right]$		
$A_4 \cosh 4\pi + 4\pi B_4 \sinh 4\pi = \left(\frac{a}{R}\right)^2 \frac{1}{8} \left[\frac{\frac{1}{4}\pi^2}{4} \left(\frac{1}{4} + \frac{1}{25} \right) \right]$		
$(A_2 + B_2) \sinh 2\pi + 2\pi B_2 \cosh 2\pi = 0$	from shear stress consideration	
$(A_4 + B_4) \sinh 4\pi + 4\pi B_4 \cosh 4\pi = 0$		

$$\frac{\phi_2}{aE^2} = -\frac{\sigma}{E} + \left(\frac{a^2}{R}\right) \frac{f}{8} \left[\left\{ \frac{\frac{1}{16}\pi^2}{4} + \frac{1}{4} \left(\frac{\frac{1}{2}\pi^2}{2} - 1 \right) \cos \frac{2\pi y}{a} + \frac{\frac{1}{100}\pi^2}{100} \cos \frac{4\pi y}{a} \right\} - \left\{ a_2 \cosh \left(\frac{2\pi y}{a} \right) + b_2 \left(\frac{2\pi y}{a} \right) \sinh \left(\frac{2\pi y}{a} \right) \right\} \right] \cos \frac{2\pi x}{a}$$

$$+ \left(\frac{a^2}{R}\right) \frac{f}{8} \left[\left\{ \frac{\frac{1}{16}\pi^2}{16} + \frac{\frac{1}{25}\pi^2}{25} \cos \frac{2\pi y}{a} \right\} - \left\{ a_4 \cosh \left(\frac{4\pi y}{a} \right) + b_4 \left(\frac{4\pi y}{a} \right) \sinh \left(\frac{4\pi y}{a} \right) \right\} \right] \cos \frac{4\pi x}{a}$$

$$\frac{1}{aE^2} \int_0^a \int_0^a \phi_y^2 dy = \left(\frac{\sigma}{E} \right)^2 + \left(\frac{a^2}{R} \right) \frac{f^2}{128} \left[\left\{ \frac{\frac{1}{4}\pi^2}{4} + \frac{1}{4} \left(\frac{\frac{1}{2}\pi^2}{2} - 1 \right) \cos \frac{2\pi y}{a} + \frac{\frac{1}{100}\pi^2}{100} \cos \frac{4\pi y}{a} \right\} - \left\{ a_2 \cosh \left(\frac{2\pi y}{a} \right) + b_2 \left(\frac{2\pi y}{a} \right) \sinh \left(\frac{2\pi y}{a} \right) \right\} \right]^2$$

$$+ \left(\frac{a^2}{R} \right) \frac{f^2}{128} \left[\left\{ \frac{\frac{1}{16}\pi^2}{16} + \frac{\frac{1}{25}\pi^2}{25} \cos \frac{2\pi y}{a} \right\} - \left\{ a_4 \cosh \left(\frac{4\pi y}{a} \right) + b_4 \left(\frac{4\pi y}{a} \right) \sinh \left(\frac{4\pi y}{a} \right) \right\} \right]^2$$

$$\frac{1}{aE^2} \int_0^a \int_0^a \phi_y^2 dx dy = \left(\frac{\sigma}{E} \right)^2 + \left(\frac{a^2}{R} \right) \frac{f^2}{256} \left[2 \left(\frac{\frac{1}{4}\pi^2}{4} \right)^2 + \frac{1}{16} \left(\frac{\frac{1}{2}\pi^2}{2} - 1 \right)^2 + \left(\frac{\frac{1}{100}\pi^2}{100} \right)^2 + \left(\frac{\frac{1}{25}\pi^2}{25} \right)^2 \right]$$

$$- \left(\frac{a^2}{R} \right) \frac{f^2}{64} \left[\int_0^1 \left\{ \frac{\frac{1}{4}\pi^2}{4} + \frac{1}{4} \left(\frac{\frac{1}{2}\pi^2}{2} - 1 \right) \cos \frac{2\pi y}{a} + \frac{\frac{1}{100}\pi^2}{100} \cos \frac{4\pi y}{a} \right\} \left\{ a_2 \cosh \left(\frac{2\pi y}{a} \right) + b_2 \left(\frac{2\pi y}{a} \right) \sinh \left(\frac{2\pi y}{a} \right) \right\} d \left(\frac{y}{a} \right) \right.$$

$$\left. + \int_0^1 \left\{ \frac{\frac{1}{16}\pi^2}{16} + \frac{\frac{1}{25}\pi^2}{25} \cos \frac{2\pi y}{a} \right\} \left\{ a_4 \cosh \left(\frac{4\pi y}{a} \right) + b_4 \left(\frac{4\pi y}{a} \right) \sinh \left(\frac{4\pi y}{a} \right) \right\} d \left(\frac{y}{a} \right) \right]$$

$$+ \left(\frac{a^2}{R} \right) \frac{f^2}{128} \left[\int_0^1 \left\{ a_2 \cosh \left(\frac{2\pi y}{a} \right) + b_2 \left(\frac{2\pi y}{a} \right) \sinh \left(\frac{2\pi y}{a} \right) \right\}^2 d \left(\frac{y}{a} \right) + \int_0^1 \left\{ a_4 \cosh \left(\frac{4\pi y}{a} \right) + b_4 \left(\frac{4\pi y}{a} \right) \sinh \left(\frac{4\pi y}{a} \right) \right\}^2 d \left(\frac{y}{a} \right) \right]$$

$$\begin{aligned}
 \int_0^1 \left\{ a_2 \cosh\left(\frac{2\pi y}{a}\right) + b_2 \left(\frac{2\pi y}{a}\right) \sinh\left(\frac{2\pi y}{a}\right) \right\} d\left(\frac{y}{a}\right) &= \frac{1}{2\pi} \int_0^1 \left\{ a_2 \cosh\left(\frac{2\pi y}{a}\right) + b_2 \left(\frac{2\pi y}{a}\right) \sinh\left(\frac{2\pi y}{a}\right) \right\} d\left(\frac{2\pi y}{a}\right) \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \left\{ a_2 \cosh u + b_2 u \sinh u \right\} du = \frac{1}{2\pi} \left[a_2 \sinh u + b_2 (u \cosh u - \sinh u) \right]_0^{2\pi} \\
 &= \frac{1}{2\pi} \left[(a_2 - b_2) \sinh 2\pi + \frac{1}{2} 2\pi \cosh 2\pi \right]
 \end{aligned}$$

$$\begin{aligned}
 \int_0^1 \cosh \frac{2\pi y}{a} \left\{ a_2 \cosh\left(\frac{2\pi y}{a}\right) + b_2 \left(\frac{2\pi y}{a}\right) \sinh\left(\frac{2\pi y}{a}\right) \right\} d\left(\frac{y}{a}\right) \\
 &= a_2 \int_0^1 \frac{1}{2} \left\{ \cosh \frac{2\pi y}{a} (y + iy) + \cosh \frac{2\pi y}{a} (y - iy) \right\} d\left(\frac{y}{a}\right) + b_2 \int_0^1 \frac{1}{2} \left(\frac{2\pi y}{a}\right) \left\{ \sinh \frac{2\pi y}{a} (y + iy) + \sinh \frac{2\pi y}{a} (y - iy) \right\} d\left(\frac{y}{a}\right) \\
 &= \frac{a_2}{2} \frac{1}{2\pi} \left[\frac{1}{1+i} \sinh \frac{2\pi y}{a} (y + iy) + \frac{1}{1-i} \sinh \frac{2\pi y}{a} (y - iy) \right]_0^1 \\
 &\quad + \frac{b_2}{2} \frac{1}{2\pi} \left[\frac{1}{(1+i)^2} \left\{ \frac{2\pi y}{a} (1+i) \cosh \frac{2\pi y}{a} (y + iy) - \sinh \frac{2\pi y}{a} (y + iy) \right\} \right. \\
 &\quad \left. + \frac{1}{(1-i)^2} \left\{ \frac{2\pi y}{a} (1-i) \cosh \frac{2\pi y}{a} (y - iy) - \sinh \frac{2\pi y}{a} (y - iy) \right\} \right]_0^1
 \end{aligned}$$

$$\begin{aligned}
& \int_0^1 \cos \frac{2\pi x}{a} \left\{ a_2 \cosh \frac{2\pi x}{a} + b_2 \left(\frac{2\pi x}{a} \right) \sinh \left(\frac{2\pi x}{a} \right) \right\} d\left(\frac{x}{a}\right) \\
&= \frac{a_2}{2} \frac{1}{2\pi} \left[\sinh \frac{2\pi x}{a} \cos \frac{2\pi x}{a} + \cosh \frac{2\pi x}{a} \sin \frac{2\pi x}{a} \right]_0^1 \\
&+ \frac{b_2}{2} \frac{1}{2\pi} \left[\left(\frac{2\pi x}{a} \right) \left\{ \cosh \frac{2\pi x}{a} \cos \frac{2\pi x}{a} + \sinh \frac{2\pi x}{a} \sin \frac{2\pi x}{a} \right\} - \cosh \frac{2\pi x}{a} \sin \frac{2\pi x}{a} \right]_0^1 \\
&= \frac{a_2}{4\pi} \left[\sinh 2\pi \right] + \frac{b_2}{4\pi} \left[2\pi \cosh 2\pi \right]
\end{aligned}$$

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$$\begin{aligned}
& \int_0^1 \cos \frac{4\pi x}{a} \left\{ a_2 \cosh \frac{2\pi x}{a} + b_2 \left(\frac{2\pi x}{a} \right) \sinh \left(\frac{2\pi x}{a} \right) \right\} d\left(\frac{x}{a}\right) \\
&= \frac{a_2}{4\pi} \left[\frac{1}{1+2i} \sinh \frac{2\pi x}{a} (1+2i) + \frac{1}{1-2i} \sinh \frac{2\pi x}{a} (1-2i) \right]_0^1 \\
&+ \frac{b_2}{4\pi} \left[\frac{1}{(1+2i)^2} \left\{ \frac{2\pi x}{a} (1+2i) \cosh \frac{2\pi x}{a} (1+2i) - \sinh \frac{2\pi x}{a} (1+2i) \right\} \right. \\
&\quad \left. + \frac{1}{(1-2i)^2} \left\{ \frac{2\pi x}{a} (1-2i) \cosh \frac{2\pi x}{a} (1-2i) - \sinh \frac{2\pi x}{a} (1-2i) \right\} \right]_0^1 \\
&- \frac{a_2}{4\pi} \left[\frac{2}{5} \sinh \frac{2\pi x}{a} \cos \frac{4\pi x}{a} + \frac{4}{5} \cosh \frac{2\pi x}{a} \sin \frac{4\pi x}{a} \right]_0^1 \\
&+ \frac{b_2}{4\pi} \left[\left(\frac{2\pi x}{a} \right) \left\{ \frac{2}{5} \cosh \frac{2\pi x}{a} \cos \frac{4\pi x}{a} + \frac{4}{5} \sinh \frac{2\pi x}{a} \sin \frac{4\pi x}{a} \right\} + \left\{ \frac{6}{25} \sinh \frac{2\pi x}{a} \cos \frac{4\pi x}{a} - \frac{6}{25} \cosh \frac{2\pi x}{a} \sin \frac{4\pi x}{a} \right\} \right]_0^1
\end{aligned}$$

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$$\int_0^1 \cos \frac{4\pi x}{a} \left\{ a_2 \cosh \frac{4\pi x}{a} + b_2 \left(\frac{2\pi x}{a} \right) \sinh \left(\frac{2\pi x}{a} \right) d\left(\frac{x}{a}\right) \right.$$

$$= \frac{a_2}{4\pi} \frac{2}{5} \sinh 2\pi + \frac{b_2}{4\pi} \left[2\pi \frac{2}{5} \cosh 2\pi + \frac{6}{25} \sinh 2\pi \right]$$

$$\int_0^1 \left\{ a_4 \cosh \frac{4\pi x}{a} + b_4 \left(\frac{4\pi x}{a} \right) \sinh \frac{4\pi x}{a} \right\} d\left(\frac{x}{a}\right) = \frac{1}{4\pi} \left[(a_4 - b_4) \sinh 4\pi + b_4 4\pi \cosh 4\pi \right]$$

$$\int_0^1 \cos \frac{2\pi x}{a} \left\{ a_4 \cosh \left(\frac{4\pi x}{a} \right) + b_4 \left(\frac{4\pi x}{a} \right) \sinh \frac{4\pi x}{a} \right\} d\left(\frac{x}{a}\right)$$

$$= \frac{a_4}{2} \int_0^1 \left[\cosh \frac{2\pi x}{a} (2+i) + \cosh \frac{2\pi x}{a} (2-i) \right] d\left(\frac{x}{a}\right) + \frac{b_4}{2} \int_0^1 \left(\frac{2\pi x}{a} \right) \left[\sinh \frac{2\pi x}{a} (2+i) + \sinh \frac{2\pi x}{a} (2-i) \right] d\left(\frac{x}{a}\right)$$

$$= \frac{a_4}{2} \frac{1}{2\pi} \left[\frac{1}{2+i} \sinh \frac{2\pi x}{a} (2+i) + \frac{1}{2-i} \sinh \frac{2\pi x}{a} (2-i) \right]'_0 + \frac{b_4}{2\pi} \left[\frac{1}{(2+i)^2} \left\{ \frac{2\pi x}{a} (2+i) \cosh \frac{2\pi x}{a} (2+i) - \sinh \frac{2\pi x}{a} (2+i) \right\} \right]'$$

$$+ \frac{1}{(2-i)^2} \left\{ \frac{2\pi x}{a} (2-i) \cosh \frac{2\pi x}{a} (2-i) - \sinh \frac{2\pi x}{a} (2-i) \right\}'_0$$

$$= \frac{a_4}{4\pi} \left[\frac{4}{5} \sinh \frac{4\pi x}{a} \cosh \frac{2\pi x}{a} + \frac{2}{5} \cosh \frac{4\pi x}{a} \sinh \frac{2\pi x}{a} \right]'_0 + \frac{b_4}{2\pi} \left[\left(\frac{2\pi x}{a} \right) \frac{4}{5} \cosh \frac{4\pi x}{a} \cosh \frac{2\pi x}{a} \right]'_0$$

$$+ \left(\frac{2\pi x}{a} \right) \frac{2}{5} \sinh \frac{4\pi x}{a} \sinh \frac{2\pi x}{a} - \left\{ \frac{6}{25} \sinh \frac{4\pi x}{a} \cosh \frac{2\pi x}{a} + \frac{2}{25} \cosh \frac{4\pi x}{a} \sinh \frac{2\pi x}{a} \right\}'_0$$

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$$\int_0^1 \cos \frac{2\pi x}{a} \left\{ a_4 \cosh \left(\frac{4\pi x}{a} \right) + b_4 \left(\frac{4\pi x}{a} \right) \sinh \frac{4\pi x}{a} \right\} d \left(\frac{x}{a} \right) \\ = \frac{a_4}{4\pi} \frac{4}{5} \sinh 4\pi + \frac{b_4}{2\pi} \left[2\pi \frac{4}{5} \cosh 4\pi - \frac{6}{25} \sinh 4\pi \right]$$

$$\int_0^1 \left\{ a_2 \cosh \left(\frac{2\pi x}{a} \right) + b_2 \left(\frac{2\pi x}{a} \right) \sinh \left(\frac{2\pi x}{a} \right) \right\}^2 d \left(\frac{x}{a} \right) \\ = \frac{a_2^2}{2} \int_0^1 \left(\cosh \frac{4\pi x}{a} + 1 \right) d \left(\frac{x}{a} \right) + a_2 b_2 \int_0^1 \left(\frac{2\pi x}{a} \right) \sinh \frac{4\pi x}{a} d \left(\frac{x}{a} \right) + \frac{b_2^2}{2} \int_0^1 \left(\frac{2\pi x}{a} \right)^2 \left[\cosh \frac{4\pi x}{a} - 1 \right] d \left(\frac{x}{a} \right) \\ = \frac{a_2^2}{2} \left[\frac{1}{4\pi} \sinh \frac{4\pi x}{a} + \frac{x}{a} \right]_0^1 + \frac{a_2 b_2}{8\pi} \left[\left(\frac{4\pi x}{a} \right) \cosh \frac{4\pi x}{a} - \sinh \frac{4\pi x}{a} \right]_0^1 \\ + \frac{b_2^2}{2} \left[\frac{1}{16\pi} \left\{ \left(\frac{4\pi x}{a} \right)^2 + 2 \right\} \sinh \frac{4\pi x}{a} - \frac{8\pi x}{a} \cosh \frac{4\pi x}{a} \right] - \frac{4\pi^2}{3} \left(\frac{x}{a} \right)^3 \Big]_0^1 \\ = \frac{a_2^2}{2} \left[\frac{1}{4\pi} \sinh 4\pi + 1 \right] + \frac{a_2 b_2}{8\pi} \left[4\pi \cosh 4\pi - \sinh 4\pi \right] \\ + \frac{b_2^2}{2} \left[\frac{1}{16\pi} \left\{ (16\pi^2 + 2) \sinh 4\pi - 8\pi \cosh 4\pi \right\} - \frac{4\pi^2}{3} \right]$$

$$\begin{aligned}
& \int_0^1 \left\{ a_4 \cosh\left(\frac{4\pi y}{a}\right) + b_4 \left(\frac{4\pi y}{a}\right) \sinh\left(\frac{4\pi y}{a}\right) \right\}^2 dy \\
&= \frac{a_4^2}{2} \left[\frac{1}{8\pi} \sinh 8\pi + 1 \right] + \frac{a_4 b_4}{16\pi} \left[16 \cosh 8\pi - \sinh 8\pi \right] + \frac{b_4^2}{2} \left[\frac{1}{32\pi} \left\{ (64\pi^2 + 2) \sinh 8\pi - 16\pi \cosh 8\pi \right\} \right. \\
&\quad \left. - \frac{16\pi^2}{3} \right]
\end{aligned}$$

$$\begin{aligned}
\frac{1}{a^2 E^2} \int_0^a \int_0^a dx dy &= \left(\frac{a}{R}\right)^4 \frac{1}{256} \left[\frac{1}{8} \left(\frac{1\pi^2}{2}\right)^2 + \frac{1}{16} \left(\frac{1\pi^2}{2} - 1\right)^2 + \frac{1\pi^2}{1000} + \frac{1}{128} (1\pi^2)^2 + \frac{1}{625} (1\pi^2)^2 \right] \\
&- \left(\frac{a}{R}\right)^4 \frac{1\pi^2}{64} \left[\frac{1\pi^2}{4} \left\{ \frac{(a_2 - b_2)}{2\pi} \sinh 2\pi + b_2 \cosh 2\pi \right\} + \frac{1}{4} \left(\frac{1\pi^2}{2} - 1\right) \left\{ a_2 \frac{\sinh 2\pi}{4\pi} + b_2 \frac{\cosh 2\pi}{2} \right\} \right. \\
&+ \frac{1\pi^2}{160} \left\{ \frac{1}{10\pi} a_2 \sinh 2\pi + b_2 \left(-\frac{1}{5} \cosh 2\pi + \frac{3}{10\pi} \sinh 2\pi\right) \right\} \\
&+ \frac{1\pi^2}{16} \left\{ \frac{(a_4 - b_4)}{4\pi} \sinh 4\pi + b_4 \cosh 4\pi + \frac{1\pi^2}{25} \left\{ \frac{1}{5\pi} a_4 \sinh 4\pi + b_4 \left(\frac{1}{5} \cosh 4\pi - \frac{3}{25\pi} \sinh 4\pi\right) \right\} \right\} \\
&+ \left(\frac{a}{R}\right)^2 \frac{1\pi^2}{128} \left[\frac{a_2^2}{2} \left(\frac{\sinh 4\pi}{4\pi} + 1 \right) + \frac{a_2 b_2}{2} \left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi} \right) + \frac{b_2^2}{2} \left(\frac{16\pi^2 + 2}{16\pi} \sinh 4\pi - \frac{1}{2} \cosh 4\pi - \frac{4\pi^2}{3} \right) \right. \\
&+ \frac{a_4^2}{2} \left(\frac{\sinh 8\pi}{8\pi} + 1 \right) + \frac{a_4 b_4}{2} \left(\cosh 8\pi - \frac{\sinh 8\pi}{8\pi} \right) + \frac{b_4^2}{2} \left(\frac{64\pi^2 + 2}{32\pi} \sinh 8\pi - \frac{1}{2} \cosh 8\pi - \frac{16\pi^2}{3} \right) \left. \right]
\end{aligned}$$

$$\frac{1}{E} \sigma_x = \left(\frac{d}{R}\right)^2 \frac{1}{8} \left[\left\{ (a_2 + 2b_2) \cosh\left(\frac{2\pi x}{a}\right) + b_2 \left(\frac{2\pi x}{a}\right) \sinh\left(\frac{2\pi x}{a}\right) \right\} \cos\frac{2\pi x}{a} \right.$$

$$+ \left\{ (a_4 + 2b_4) \cosh\left(\frac{4\pi x}{a}\right) + b_4 \left(\frac{4\pi x}{a}\right) \sinh\left(\frac{4\pi x}{a}\right) \right\} \cos\frac{4\pi x}{a}$$

$$+ \left(\frac{1}{4} - 1 \right) \cos\frac{2\pi x}{a} + \frac{1}{4} \left(\frac{2\pi x}{a} - 1 \right) \cos\frac{2\pi x}{a} + \frac{1}{4} \left\{ \frac{1}{4} \cos\frac{4\pi x}{a} + \frac{1}{25} \cos\frac{4\pi x}{a} + \frac{1}{25} \cos\frac{4\pi x}{a} \right\}$$

$$\frac{\sigma_x}{E} = \left(\frac{d}{R}\right)^2 \frac{1}{8} \left[\left\{ \left(\frac{1}{4} - 1\right) + \frac{1}{4} \left(\frac{1}{2} - 1\right) \cos\frac{2\pi x}{a} + \frac{1}{25} \cos\frac{4\pi x}{a} + (a_2 + 2b_2) \cosh\frac{2\pi x}{a} + b_2 \left(\frac{2\pi x}{a}\right) \sinh\frac{2\pi x}{a} \right\} \cos\frac{2\pi x}{a} \right.$$

$$+ \left\{ \frac{1}{16} - 1 + \frac{1}{100} \cos\frac{2\pi x}{a} + (a_4 + 2b_4) \cosh\frac{4\pi x}{a} + b_4 \left(\frac{4\pi x}{a}\right) \sinh\frac{4\pi x}{a} \right\} \cos\frac{4\pi x}{a}$$

$$\frac{1}{E} \int_0^a \sigma_x^2 dx = \left(\frac{d}{R}\right)^4 \frac{1}{256} \left[2 \left(\frac{1}{4} - 1 \right)^2 + \frac{1}{16} \left(\frac{1}{2} - 1 \right)^2 + 2 \left(\frac{1}{16} \right)^2 + \frac{1}{10,000} + \left(\frac{1}{25} \right)^2 \right]$$

$$+ \left(\frac{d}{R}\right)^4 \frac{1}{64} \left[\int_0^1 \left\{ \left(\frac{1}{4} - 1\right) + \frac{1}{4} \left(\frac{1}{2} - 1\right) \cos\frac{2\pi x}{a} + \frac{1}{25} \cos\frac{4\pi x}{a} + (a_2 + 2b_2) \cosh\frac{2\pi x}{a} + b_2 \left(\frac{2\pi x}{a}\right) \sinh\frac{2\pi x}{a} \right\} d\left(\frac{x}{a}\right) \right.$$

$$+ \int_0^1 \left\{ \frac{1}{16} - 1 + \frac{1}{100} \cos\frac{2\pi x}{a} + (a_4 + 2b_4) \cosh\frac{4\pi x}{a} + b_4 \left(\frac{4\pi x}{a}\right) \sinh\frac{4\pi x}{a} \right\} d\left(\frac{x}{a}\right) \Big]$$

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$$\begin{aligned}
\frac{1}{a^2 E^2} \int_0^a \int_0^a \psi^2 dx dy &= \left(\frac{a}{R}\right)^2 \frac{f^2}{256} \left[2 \left(\frac{f^2}{4} - 1\right)^2 + \frac{1}{16} \left(\frac{f^2}{2} - 1\right)^2 + \frac{1}{128} (f\pi)^2 + \frac{f^2 \pi^2}{10,000} + \left(\frac{f\pi}{25}\right)^2 \right] \\
&+ \left(\frac{a}{R}\right)^2 \frac{f^2}{64} \left[\left(\frac{f^2}{4} - 1\right) \left\{ \frac{(a_2 + b_2)}{2\pi} \sin 2\pi + b_2 \cos 2\pi \right\} + \frac{1}{4} \left(\frac{f^2}{2} - 1\right) (a_2 + 2b_2) \frac{\sin 2\pi}{4\pi} + b_2 \frac{\cos 2\pi}{2} \right] \\
&+ \frac{f^2}{25} \left\{ \frac{1}{10\pi} (a_2 + 2b_2) \sin 2\pi + b_2 \left(\frac{1}{5} \cos 2\pi + \frac{2}{50\pi} \sin 2\pi\right) \right\} \\
&+ \frac{f^2}{16} \left\{ \frac{(a_4 + b_4)}{4\pi} \sin 4\pi + b_4 \cos 4\pi \right\} + \frac{f^2}{100} \left\{ \frac{1}{5\pi} (a_4 + 2b_4) \sin 4\pi + b_4 \left(\frac{4}{5} \cos 4\pi - \frac{2}{25\pi} \sin 4\pi\right) \right\} \\
&+ \left(\frac{a}{R}\right)^2 \frac{f^2}{128} \left[\frac{(a_2 + 2b_2)^2}{2} \left(\frac{\sin 6\pi}{4\pi} + 1 \right) + \frac{(a_2 + 2b_2)b_2}{9} \left(\cos 4\pi - \frac{\sin 4\pi}{4\pi} \right) + \frac{b_2^2}{2} \left(\frac{16\pi^2}{16\pi} \sin 4\pi - \frac{1}{2} \cos 4\pi - \frac{4\pi^2}{3} \right) \right] \\
&+ \frac{(a_4 + 2b_4)^2}{2} \left(\frac{\sin 8\pi}{8\pi} + 1 \right) + \frac{(a_4 + 2b_4)b_4}{9} \left(\cos 8\pi - \frac{\sin 8\pi}{8\pi} \right) + \frac{b_4^2}{2} \left(\frac{64\pi^2}{32\pi} \sin 8\pi - \frac{1}{2} \cos 8\pi - \frac{16\pi^2}{3} \right) \right]
\end{aligned}$$

$$\frac{1}{E} \bar{u}_y = \left(\frac{q}{R}\right) \frac{f}{8} \left[\frac{1}{4} \left(\frac{f^2}{2} - 1\right) \sin \frac{2\pi y}{a} \sin \frac{2\pi x}{a} + \frac{f^2}{4} \left\{ \frac{2}{25} \sin \frac{4\pi x}{a} \sin \frac{2\pi y}{a} + \frac{2}{25} \sin \frac{2\pi x}{a} \sin \frac{4\pi y}{a} \right\} \right. \\ \left. + \left\{ (a_2 + b_2) \sinh \frac{2\pi y}{a} + b_2 \left(\frac{2\pi y}{a}\right) \cosh \frac{2\pi y}{a} \right\} \sin \frac{2\pi x}{a} + \left\{ (a_4 + b_4) \sinh \frac{4\pi y}{a} + b_4 \left(\frac{4\pi y}{a}\right) \cosh \frac{4\pi y}{a} \right\} \sin \frac{4\pi x}{a} \right]$$

$$\frac{1}{aE^2} \int_0^a \int_0^a \bar{u}_{xy}^2 dx dy = \left(\frac{q}{R}\right)^4 \frac{f^2}{128} \left[\left\{ \frac{1}{4} \left(\frac{f^2}{2} - 1\right) \sin \frac{2\pi y}{a} + (a_2 + b_2) \sinh \frac{2\pi y}{a} + b_2 \left(\frac{2\pi y}{a}\right) \cosh \frac{2\pi y}{a} \right\}^2 \right. \\ \left. + \left\{ \frac{f^2}{50} \sin \frac{4\pi y}{a} + (a_4 + b_4) \sinh \frac{4\pi y}{a} + b_4 \left(\frac{4\pi y}{a}\right) \cosh \frac{4\pi y}{a} \right\}^2 \right]$$

$$\frac{1}{aE^2} \int_0^a \int_0^a \bar{u}_{xy}^2 dx dy = \left(\frac{q}{R}\right)^4 \frac{f^2}{256} \left[\frac{1}{16} \left(\frac{f^2}{2} - 1\right)^2 + 2 \left(\frac{f^2}{50}\right)^2 \right] \\ + \left(\frac{q}{R}\right)^4 \frac{f^2}{64} \left[\int_0^1 \left\{ \frac{1}{4} \left(\frac{f^2}{2} - 1\right) \sin \frac{2\pi y}{a} + \frac{f^2}{50} \sin \frac{4\pi y}{a} \right\} \left\{ (a_2 + b_2) \sinh \frac{2\pi y}{a} + b_2 \left(\frac{2\pi y}{a}\right) \cosh \frac{2\pi y}{a} \right\} d\left(\frac{y}{a}\right) \right. \\ \left. + \frac{1}{50} f^2 \int_0^1 \sin \frac{2\pi y}{a} \left\{ (a_4 + b_4) \sinh \frac{4\pi y}{a} + b_4 \left(\frac{4\pi y}{a}\right) \cosh \frac{4\pi y}{a} \right\} d\left(\frac{y}{a}\right) \right] \\ + \left(\frac{q}{R}\right)^4 \frac{f^2}{128} \left[\int_0^1 \left\{ (a_2 + b_2) \sinh \frac{2\pi y}{a} + b_2 \left(\frac{2\pi y}{a}\right) \cosh \frac{2\pi y}{a} \right\}^2 d\left(\frac{y}{a}\right) + \int_0^1 \left\{ (a_4 + b_4) \sinh \frac{4\pi y}{a} + b_4 \left(\frac{4\pi y}{a}\right) \cosh \frac{4\pi y}{a} \right\}^2 d\left(\frac{y}{a}\right) \right]$$

$$\begin{aligned}
 \int_0^1 \sin \frac{2\pi x}{a} \sinh \frac{2\pi y}{a} d\left(\frac{y}{a}\right) &= \frac{1}{2i} \int_0^1 \left[\cosh \frac{2\pi y}{a} (1+i) - \cosh \frac{2\pi y}{a} (1-i) \right] d\left(\frac{y}{a}\right) \\
 &= \frac{1}{4\pi i} \left[\frac{1}{1+i} \sinh \frac{2\pi y}{a} (1+i) - \frac{1}{1-i} \sinh \frac{2\pi y}{a} (1-i) \right]_0^1 = \frac{1}{4\pi} \left[-\sinh \frac{2\pi y}{a} \cos \frac{2\pi y}{a} + \cosh \frac{2\pi y}{a} \sin \frac{2\pi y}{a} \right]_0^1 \\
 &= -\frac{\sinh 2\pi}{4\pi}
 \end{aligned}$$

$$\begin{aligned}
 \int_0^1 \sin \frac{2\pi x}{a} \left(\frac{2\pi y}{a}\right) \cosh \frac{2\pi y}{a} d\left(\frac{y}{a}\right) &= \frac{1}{2i} \int_0^1 \left(\frac{2\pi y}{a}\right) \left[\sinh \frac{2\pi y}{a} (1+i) - \sinh \frac{2\pi y}{a} (1-i) \right] d\left(\frac{y}{a}\right) \\
 &= \frac{1}{4\pi i} \int_0^1 \left[\frac{1}{(1+i)^2} \left\{ \frac{2\pi y}{a} (1+i) \cosh \frac{2\pi y}{a} (1+i) - \sinh \frac{2\pi y}{a} (1+i) \right\} - \frac{1}{(1-i)^2} \left\{ \frac{2\pi y}{a} (1-i) \cosh \frac{2\pi y}{a} (1-i) - \sinh \frac{2\pi y}{a} (1-i) \right\} \right] d\left(\frac{y}{a}\right) \\
 &= \frac{1}{4\pi} \int_0^1 \left[\frac{2\pi y}{a} \left\{ -\cosh \frac{2\pi y}{a} \cos \frac{2\pi y}{a} + \sinh \frac{2\pi y}{a} \sin \frac{2\pi y}{a} \right\} + \sinh \frac{2\pi y}{a} \cos \frac{2\pi y}{a} \right] d\left(\frac{y}{a}\right) \\
 &= \left[-\frac{1}{2} \cosh 2\pi + \frac{\sinh 2\pi}{4\pi} \right]
 \end{aligned}$$

$$\begin{aligned}
 \int_0^1 \sin \frac{4\pi x}{a} \sinh \frac{2\pi y}{a} d\left(\frac{y}{a}\right) &= \frac{1}{4\pi i} \left[\frac{1}{1+2i} \sinh \frac{2\pi y}{a} (1+2i) - \frac{1}{1-2i} \sinh \frac{2\pi y}{a} (1-2i) \right]_0^1 \\
 &= \frac{1}{4\pi} \left[-\frac{4}{5} \sinh \frac{2\pi y}{a} \cos \frac{4\pi y}{a} + \frac{2}{5} \cosh \frac{2\pi y}{a} \sin \frac{4\pi y}{a} \right]_0^1 = -\frac{\sinh 2\pi}{5\pi}
 \end{aligned}$$

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$$\int_0^1 \sin \frac{2\pi x}{a} \sinh \frac{4\pi x}{a} d\left(\frac{x}{a}\right) = \frac{1}{4\pi i} \left[\frac{1}{2+i} \sinh \frac{2\pi x}{a} (2+i) - \frac{1}{2-i} \sinh \frac{2\pi x}{a} (2-i) \right]'_0$$

$$= \frac{1}{4\pi} \left[-\frac{2}{5} \sinh \frac{4\pi x}{a} \cos \frac{2\pi x}{a} + \frac{4}{5} \cosh \frac{4\pi x}{a} \sin \frac{2\pi x}{a} \right]'_0 = -\frac{\sinh 6\pi}{10\pi}$$

$$\int_0^1 \frac{4\pi x}{a} \sin \frac{2\pi x}{a} \cosh \frac{4\pi x}{a} d\left(\frac{x}{a}\right) = \frac{1}{2\pi i} \left[\frac{1}{(2+i)^2} \left\{ \frac{2\pi x}{a} (2+i) \cosh \frac{2\pi x}{a} (2+i) - \sinh \frac{2\pi x}{a} (2+i) \right\} \right. \\ \left. - \frac{1}{(2-i)^2} \left\{ \frac{2\pi x}{a} (2-i) \cosh \frac{2\pi x}{a} (2-i) - \sinh \frac{2\pi x}{a} (2-i) \right\} \right]'_0$$

$$= \frac{1}{2\pi} \left[\frac{2\pi x}{a} \left\{ -\frac{2}{5} \cosh \frac{4\pi x}{a} \cos \frac{2\pi x}{a} + \frac{4}{5} \sinh \frac{4\pi x}{a} \sin \frac{2\pi x}{a} \right\} + \left\{ \frac{8}{25} \sinh \frac{4\pi x}{a} \cos \frac{2\pi x}{a} \right. \right. \\ \left. \left. - \frac{6}{25} \cosh \frac{4\pi x}{a} \sin \frac{2\pi x}{a} \right\} \right]'_0$$

$$= -\frac{2}{5} \cosh 4\pi + \frac{4}{25\pi} \sinh 4\pi$$

$$\int_0^1 \left\{ (a_2 + b_2) \sinh \frac{2\pi x}{a} + b_2 \left(\frac{2\pi x}{a} \right) \cosh \frac{2\pi x}{a} \right\}^2 d\left(\frac{x}{a}\right)$$

$$= \frac{(a_2 + b_2)^2}{2\pi} \left\{ \frac{\sinh 4\pi}{4} - \pi \right\} + \frac{b_2 (a_2 + b_2)}{8a} \left\{ 4\pi \cosh 4\pi - \sinh 4\pi \right\} + \frac{b_2^2}{32\pi} \left[\frac{(4\pi)^3}{3} + (16\pi^2 + 2) \sinh 4\pi - 8\pi \cosh 4\pi \right]$$

$$\int_0^1 \left\{ (a_4 + b_4) \sinh \frac{4\pi y}{a} + b_4 \left(\frac{4\pi y}{a} \right) \cosh \left(\frac{4\pi y}{a} \right) \right\} d\left(\frac{y}{a}\right)$$

$$= \frac{(a_4 + b_4)^2}{4\pi} \left\{ \frac{\sinh 8\pi}{4} - 2\pi \right\} + \frac{b_4(a_4 + b_4)}{16\pi} \left\{ 8\pi \cosh 8\pi - \sinh 8\pi \right\} + \frac{b_4^2}{64\pi} \left[\frac{(4\pi)^3}{3} + (64\pi^2 + 2) \sinh 8\pi - 16\pi \cosh 8\pi \right]$$

$$\frac{1}{a^2 E} \int_0^a \int_0^a x y \, dx \, dy = \left(\frac{a}{R} \right)^4 \frac{1}{256} \left[\frac{1}{16} \left(\frac{1}{3} \pi^2 - 1 \right)^2 + 2 \left(\frac{1}{5} \pi^2 \right)^2 \right]$$

$$+ \left(\frac{a}{R} \right)^4 \frac{1}{64} \left[\frac{1}{4} \left(\frac{1}{2} \pi^2 - 1 \right) \left\{ -(a_2 + b_2) \frac{\sinh 2\pi}{4\pi} + b_2 \left(\frac{\sinh 2\pi}{4\pi} - \frac{1}{2} \cosh 2\pi \right) \right\} \right.$$

$$+ \frac{1}{50} \left\{ -(a_2 + b_2) \frac{\sinh 2\pi}{5\pi} + b_2 \left(-\frac{2}{5} \cosh 2\pi + \frac{2}{25\pi} \sinh 2\pi \right) \right\}$$

$$+ \frac{1}{50} \left\{ -(a_4 + b_4) \frac{\sinh 4\pi}{10\pi} + b_4 \left(-\frac{2}{5} \cosh 4\pi + \frac{4}{25\pi} \sinh 4\pi \right) \right\} \Bigg]$$

$$+ \left(\frac{a}{R} \right)^4 \frac{1}{128} \left[\frac{(a_2 + b_2)^2}{2} \left\{ \frac{\sinh 4\pi}{4\pi} - 1 \right\} + \frac{b_2(a_2 + b_2)}{2} \left\{ \cosh 4\pi - \frac{\sinh 4\pi}{4\pi} \right\} + \frac{b_2^2}{2} \left\{ \frac{4\pi^2}{3} + \frac{16\pi^2}{16\pi} \sinh 4\pi - \frac{\cosh 4\pi}{2} \right\} \right.$$

$$+ \frac{(a_4 + b_4)^2}{2} \left\{ \frac{\sinh 8\pi}{8\pi} - 1 \right\} + \frac{b_4(a_4 + b_4)}{2} \left\{ \cosh 8\pi - \frac{\sinh 8\pi}{8\pi} \right\} + \frac{b_4^2}{2} \left\{ \frac{16\pi^2}{3} + \frac{64\pi^2 + 2}{32\pi} \sinh 8\pi - \frac{\cosh 8\pi}{2} \right\} \Bigg]$$

$$\begin{aligned} \frac{1}{2} \left\{ \left(\frac{\partial w}{\partial x} \right)^2 \right\} &= \frac{1}{2} \left(\frac{a}{R} \right)^4 \left[\frac{1}{4} (1 + \cos \frac{2\pi x}{a}) \left(\frac{\pi}{a} \right) \sin \frac{2\pi x}{a} \right]^2 \\ &= \frac{1}{2} \left(\frac{a}{R} \right)^2 \frac{1}{4} \left[\frac{1}{4} \pi^2 (1 + \cos \frac{2\pi x}{a})^2 \sin^2 \frac{2\pi x}{a} \right] = \left(\frac{a}{R} \right)^2 \frac{1}{8} \left[\frac{\pi^2}{4} \left(\frac{3}{2} + 2 \cos \frac{2\pi x}{a} + \frac{1}{2} \cos \frac{4\pi x}{a} \right) \right. \\ &\quad \left. \left(\frac{1}{2} - \frac{1}{2} \cos \frac{4\pi x}{a} \right) \right] \end{aligned}$$

$$\frac{\partial w}{\partial y} = \frac{\sigma_y}{E} - \frac{1}{2} \left\{ \left(\frac{\partial w}{\partial y} \right)^2 \right\}$$

$$+ \frac{3}{16} \pi^2 \cos \frac{4\pi x}{a}$$

$$\begin{aligned} &= -\frac{\sigma_y}{E} + \left(\frac{a}{R} \right)^2 \frac{1}{8} \left[-\frac{3}{16} \pi^2 \left\{ \frac{1}{4} \left(\frac{\pi^2}{9} - 1 \right) \cos \frac{2\pi x}{a} + \frac{2\pi^2}{100} \pi^2 \cos \frac{4\pi x}{a} \right\} - \left\{ a_2 \cos \left(\frac{2\pi x}{a} \right) + b_2 \left(\frac{2\pi x}{a} \right) \sin \left(\frac{2\pi x}{a} \right) \right\} \right. \\ &\quad \left. + \left(\frac{a}{R} \right)^2 \frac{1}{8} \left[\left(\frac{1}{25} \pi^2 \cos \frac{2\pi x}{a} + \frac{1}{16} \pi^2 \cos \frac{4\pi x}{a} \right) \right] - \left\{ a_4 \cos \frac{4\pi x}{a} + b_4 \left(\frac{4\pi x}{a} \right) \sin \frac{4\pi x}{a} \right\} \right] \cos \frac{4\pi x}{a} \end{aligned}$$

$$\cos \frac{2\pi x}{a}$$

$$\begin{aligned} \frac{w}{a} &= -\frac{\sigma_y}{E} \left(\frac{a}{R} \right) + \left(\frac{a}{R} \right)^2 \frac{1}{8} \left[-\frac{3}{16} \pi^2 \left(\frac{\pi^2}{9} \right) + \frac{3}{64} \pi^2 \sin \frac{4\pi x}{a} + \right. \\ &\quad \left\{ \frac{1}{8\pi} \left(\frac{1}{2} \pi^2 - 1 \right) \sin \frac{2\pi x}{a} + \frac{2\pi^2}{400\pi} - \pi^2 \sin \frac{4\pi x}{a} \right\} \cos \frac{2\pi x}{a} - \frac{1}{32} \left\{ (a_2 - b_2) \sin \left(\frac{2\pi x}{a} \right) + b_2 \left(\frac{2\pi x}{a} \right) \cos \left(\frac{2\pi x}{a} \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{a}{R} \right)^2 \frac{1}{8} \left[\left\{ \frac{1}{50\pi} \sin \frac{2\pi x}{a} + \frac{1}{64\pi} \sin \frac{4\pi x}{a} \right\} - \frac{1}{32\pi} \left\{ (a_4 - b_4) \sin \left(\frac{4\pi x}{a} \right) + b_4 \left(\frac{4\pi x}{a} \right) \cos \left(\frac{4\pi x}{a} \right) \right\} \right] \right\} \cos \frac{4\pi x}{a} \right. \end{aligned}$$

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At $y = \pm a$,

$$\left(\frac{\psi}{a}\right)_{a} = -\frac{\sigma}{E} + \left(\frac{q}{R}\right)^3 \frac{1}{8} \left[-\frac{3}{16} R^2 - \frac{1}{2\pi} \left\{ (a_2 - b_2) \sinh 2\pi + 2\pi b_2 \cosh 2\pi \right\} \cos \frac{2\pi x}{a} \right. \\ \left. - \frac{1}{2\pi} \left\{ (a_4 - b_4) \sinh 4\pi + 4\pi b_4 \cosh 4\pi \right\} \cos \frac{4\pi x}{a} \right]$$

the potential increase $\Delta\phi$ is

$$a \int_0^a \sigma \left(\frac{\psi}{a} \right) dx, \quad \Delta\phi = - \left(\frac{q}{R} \right)^2 \frac{1}{8} \frac{\sigma}{E} \left(\frac{3}{16} R^2 \right) \quad \left[- \left(\frac{\psi}{E} \right)^2 \text{ in } \sigma_y \text{ and term} \right]$$

$$\frac{\partial \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{R} \left[\frac{1}{2} \pi^2 \cos \frac{2\pi x}{R} \left(1 + \cos \frac{2\pi x}{R} \right) \right]$$

$$\frac{t^2}{12a^2} \int_0^a \int_0^a \left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \right)^2 dx dy = \frac{1}{12} \left(\frac{t}{R} \right)^2 \left(\frac{1}{2} \pi^2 \right)^2 \int_0^a \int_0^a \left\{ \cos \frac{2\pi x}{a} \left(1 + \cos \frac{2\pi x}{a} \right) \right\}^2 dx dy \left(\frac{1}{a} \right) \\ = \frac{1}{12} \left(\frac{t}{R} \right)^2 \left(\frac{1}{2} \pi^2 \right)^2 \frac{1}{4} (2+1) = \frac{1}{12} \left(\frac{t}{R} \right)^2 \left(\frac{1}{2} \pi^2 \right)^2 \frac{3}{4}$$

$$\frac{t^2}{12a^2} \int_0^a \int_0^a \left(\frac{\partial^2 \psi}{\partial y^2} \right)^2 dx dy = \frac{1}{12} \left(\frac{t}{R} \right)^2 \left(\frac{1}{2} \pi^2 \right)^2 \frac{3}{4}$$

$$\frac{t^2}{12a^2} \int_0^a \int_0^a \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 dx dy = \frac{1}{12} \left(\frac{t}{R} \right)^2 \left(\frac{1}{2} \pi^2 \right)^2 \frac{1}{4}$$

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$$\cosh 2\pi \cdot a_2 + 2\pi \sinh 2\pi \cdot b_2 = \frac{77}{200} (f\pi^2) - \frac{1}{4}$$

$$\sinh 2\pi \cdot a_2 + (\sinh 2\pi + 2\pi \cosh 2\pi) \cdot b_2 = 0$$

$$2(\sinh 4\pi + 4\pi) b_2 = -\sinh 2\pi \left(1 - \frac{77}{50} f\pi^2\right)$$

$$b_2 = \frac{\sinh 2\pi}{2(\sinh 4\pi + 4\pi)} \left(1 - \frac{77}{50} f\pi^2\right)$$

$$a_2 = -b_2 \frac{\sinh 2\pi + 2\pi \cosh 2\pi}{\sinh 2\pi}$$

$$a_2 = -\frac{\sinh 2\pi + 2\pi \cosh 2\pi}{2(\sinh 4\pi + 4\pi)} \left(1 - \frac{77}{50} f\pi^2\right)$$

$$\cosh 4\pi \cdot a_4 + 4\pi \sinh 4\pi \cdot b_4 = \frac{41}{400} (f\pi^2)$$

$$\sinh 4\pi \cdot a_4 + (\sinh 4\pi + 4\pi \cosh 4\pi) \cdot b_4 = 0$$

$$\frac{1}{2}(\sinh 8\pi + 8\pi) b_4 = -\frac{41}{400} (f\pi^2) \sinh 4\pi$$

$$b_4 = -\frac{\sinh 4\pi}{(\sinh 8\pi + 8\pi)} \frac{41}{200} (f\pi^2)$$

$$a_4 = +\frac{\sinh 4\pi + 4\pi \cosh 4\pi}{(\sinh 8\pi + 8\pi)} \frac{41}{200} (f\pi^2)$$

$$\text{The external energy} = \frac{1}{2E} \left[\sigma_x^2 + \sigma_y^2 + 2\tau_{xy} \right]$$

$$\begin{aligned} \frac{\text{The external energy}}{E a^2 t} &= \left(\frac{a^4}{R} \right) \frac{f^2}{64} \left[\frac{1}{32} \left(\frac{f}{a} \right)^2 + \frac{1}{2} \left(\frac{f}{a} \right)^2 - 1 \right]^2 + \frac{1}{16} \left(\frac{f}{a} \right)^2 - 1 \right]^2 + \frac{\left(\frac{f}{a} \right)^2}{2000} + \frac{1}{256} \left(\frac{f}{a} \right)^2 + \frac{3}{2500} \left(\frac{f}{a} \right)^2 \\ &+ \left(\frac{a^4}{R} \right) \frac{f^2}{64} \left[-a_2 \frac{3}{4} \frac{\sinh 2\pi}{2\pi} \left\{ 1 + \frac{9}{50} \left(\frac{f}{a} \right)^2 \right\} - b_2 \left\{ \left(\frac{f}{4} \right) \frac{\sinh 2\pi}{2\pi} + \frac{3}{4} \cosh 2\pi \right\} + \left(\frac{139}{1000} \cosh 2\pi - \frac{3/23}{5000} \frac{\sinh 2\pi}{2\pi} \right) \right] \\ &- a_4 \frac{f^2}{25} \frac{\sinh 4\pi}{4\pi} - b_4 \left\{ \cosh 4\pi - \frac{33}{8} \frac{\sinh 4\pi}{4\pi} \right\} \\ &+ \left(\frac{a^4}{R} \right) \frac{f^2}{64} \left[\left(\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi \right) a_2 b_2 + \frac{1}{2} \left\{ \left(\frac{f}{a} \right)^2 + 2 \right\} \frac{\sinh 4\pi}{4\pi} + \cosh 4\pi + 1 \right\} b_2^2 \\ &+ \left(\frac{a^4}{R} \right) \frac{f^2}{64} \left[\left(\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi \right) a_4 b_4 + \frac{1}{2} \left\{ \left(\frac{f}{a} \right)^2 + 2 \right\} \frac{\sinh 4\pi}{4\pi} + \cosh 4\pi + 1 \right\} b_4^2 \right] \\ &+ \left(\frac{a^4}{R} \right) \frac{\sinh 8\pi}{8\pi} a_4^2 + \left(\frac{\sinh 8\pi}{8\pi} + \cosh 8\pi \right) a_4 b_4 + \frac{1}{2} \left\{ \left(\frac{f}{a} \right)^2 + 2 \right\} \frac{\sinh 8\pi}{8\pi} + \cosh 8\pi + 1 \right\} b_4^2 \end{aligned}$$

$$\frac{\Delta \rho}{E a^2 t} = - \left(\frac{a^2}{R} \right) \frac{f^2}{64} \left[3\pi \frac{f}{E} \right]$$

$$\frac{\text{The bending energy}}{E a^2 t} = \frac{1}{6} \left(\frac{f}{a} \right)^2 \frac{\left(\frac{f}{a} \right)^2}{4}$$

a_2	$a_2 b_2$	b_2^2
$\frac{\sinh 4\pi}{4\pi} + 1$	$4 \frac{\sinh 4\pi}{4\pi} + 4$ $\cosh 4\pi - 2 \frac{\sinh 4\pi}{4\pi}$	$4 \frac{\sinh 4\pi}{4\pi} + 4$ $2 \cosh 4\pi - 2 \frac{\sinh 4\pi}{4\pi}$ $\frac{16\pi^2 + 2}{16\pi} \sinh 4\pi - \frac{1}{2} \cosh 4\pi - \frac{4\pi^2}{3}$
$\frac{\sinh 4\pi}{4\pi} + 1$	$\cosh 4\pi - 2 \frac{\sinh 4\pi}{4\pi}$	$\frac{16\pi^2 + 2}{16\pi} \sinh 4\pi - \frac{1}{2} \cosh 4\pi - \frac{4\pi^2}{3}$
$2 \frac{\sinh 4\pi}{4\pi} - 2$	$4 \frac{\sinh 4\pi}{4\pi} - 4$ $2 \cosh 4\pi - 2 \frac{\sinh 4\pi}{4\pi}$	$2 \frac{\sinh 4\pi}{4\pi} - 2$ $2 \cosh 4\pi - 2 \frac{\sinh 4\pi}{4\pi}$ $2 \frac{4\pi^2}{3} - \cosh 4\pi + 2 \frac{16\pi^2 + 2}{16\pi} \sinh 4\pi$
$4 \frac{\sinh 4\pi}{4\pi}$	$4 \frac{\sinh 4\pi}{4\pi} + 4 \cosh 4\pi +$	$(16\pi^2 + 4) \frac{\sinh 4\pi}{4\pi} + 2 \cosh 4\pi + 2$

$$(f\pi^2)^2 \left[\frac{1}{32} + \frac{1}{32} + \frac{1}{64} + \frac{1}{2000} + \frac{1}{256} + \frac{3}{2500} \right] = (f\pi^2)^2 [0.08203125 + 0.00125000]$$

$$+ (f\pi^2) \left[-\frac{1}{4} - \frac{1}{16} \right] = - (f\pi^2) [0.3125000]$$

$$+ \left[\frac{1}{2} + \frac{1}{16} \right] = 0.5625000$$

$$\begin{aligned} \mathcal{W}_1 = & \left(\frac{g^4}{R} \right) \frac{f^2}{64} \left[0.08328125 (f\pi^2)^2 - 0.312500 (f\pi^2) + 0.5625000 \right. \\ & - 0.75 \frac{\sinh 2\pi}{2\pi} \left\{ 1 + 0.18000 (f\pi^2) \right\} a_2 - \left\{ (4.2500 \frac{\sinh 2\pi}{2\pi} + 0.3500 \cosh 2\pi) + (0.13900 \cosh 2\pi - 0.624600 \frac{\sinh 2\pi}{2\pi}) \right\} b_2 \\ & - 0.0400 \frac{\sinh 4\pi}{4\pi} (f\pi^2) a_4 - \left\{ 0.045500 - 0.165000 \frac{\sinh 4\pi}{4\pi} \right\} (f\pi^2) b_4 \\ & + \frac{\sinh 4\pi}{4\pi} a_2^2 + \left(\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi \right) a_2 b_2 + \left\{ (4\pi^2 + 1) \frac{\sinh 4\pi}{4\pi} + 0.5 \cosh 4\pi + 0.5 \right\} b_2^2 + \\ & + \frac{\sinh 8\pi}{8\pi} a_4^2 + \left(\frac{\sinh 8\pi}{8\pi} + \cosh 8\pi \right) a_4 b_2 + \left\{ (16\pi^2 + 1) \frac{\sinh 8\pi}{8\pi} + 0.5 \cosh 8\pi + 0.5 \right\} b_4^2 \end{aligned}$$

$$\mathcal{W}_2 = - \left(\frac{g^2}{R} \right) \frac{f^2}{64} \left[3\pi^2 \frac{g}{E} \right]$$

$$\mathcal{W}_3 = \left(\frac{f}{R} \right)^2 \frac{f^2}{64} \left[\frac{16\pi^4}{5} \right]$$

$$\begin{aligned}
 a_2 &= - \frac{\left(\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi \right)}{\left(\frac{\sinh 4\pi}{4\pi} + 1 \right)} (0.25000 - 0.385000 f\pi^2) \\
 b_2 &= + \frac{\frac{\sinh 2\pi}{2\pi}}{\left(\frac{\sinh 4\pi}{4\pi} + 1 \right)} (0.25000 - 0.385000 f\pi^2) \\
 a_4 &= + \frac{\left(\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi \right)}{\left(\frac{\sinh 8\pi}{8\pi} + 1 \right)} 0.102500 (f\pi^2) \\
 b_4 &= - \frac{\left(\frac{\sinh 4\pi}{4\pi} \right)}{\left(\frac{\sinh 8\pi}{8\pi} + 1 \right)} 0.102500 (f\pi^2)
 \end{aligned}$$

$$\log_e(e^\xi) = \xi = 2.30258509 \log_{10}(e^\xi)$$

$$\log_{10}(e^\xi) = 0.434294482 \xi$$

$$\therefore \log_{10}(e^{2\pi}) = 0.434294482 \times 6.273165307 = 2.7272502$$

$$e^{2\pi} = 535.49162, \quad e^{-2\pi} = 0.00187$$

$$\sinh 2\pi = 267.744625 \quad \cosh 2\pi = 267.74462$$

$$\frac{\sinh 2\pi}{2\pi} = 42.613218 \quad \cosh 2\pi = 267.74462$$

$$\log_{10}(e^{4\pi}) = 5.45750542,$$

$$e^{4\pi} = 286751.33$$

$$\sinh 4\pi = 143375.66$$

$$\frac{\sinh 4\pi}{4\pi} = 11409.473, \quad \cosh 4\pi = 143375.66$$

$$\log_{10}(e^{8\pi}) = 10.91501083, \quad e^{8\pi} = 82226314000$$

$$\frac{\sinh 8\pi}{8\pi} = 1635840500 \quad \cosh 8\pi = 41113157000$$

$$a_2 = -0.027199735 (0.2500 - 0.38500i\pi^2)$$

$$b_2 = +0.0037345707 (0.2500 - 0.38500i\pi^2)$$

$$a_4 = +0.000094621164 (0.1025000 + i\pi^2)$$

$$b_4 = -0.0000069746655 (0.1025000 - i\pi^2)$$

$$\begin{aligned}
& + 0.08328125(f\pi^2)^2 - 0.3125000(f\pi^2) + 0.5625000 \quad \underline{476} \\
& + (0.25000 - 0.385000f\pi^2) \left\{ 0.86930118(1 + 0.1800f\pi^2) - (0.94887221 + 0.03958959f\pi^2) \right\} \\
& - 0.1025000(f\pi^2)^2 \left\{ 0.043183105 - 0.026869720 \right\} + \\
& + (0.2500 - 0.38500f\pi^2)^2 \left\{ 8.4410200 - 15.7229706 + 7.4410933 \right\} \\
& + (0.1025000f\pi^2)^2 \left\{ 14.6459494 - 28.2123232 + 13.6459506 \right\} \\
& = 0.08328125(f\pi^2)^2 - 0.3125000(f\pi^2) + 0.5625000 \\
& - (0.25000 - 0.385000f\pi^2)(0.07957103 - 0.11688462f\pi^2) - 0.001672122(f\pi^2)^2 \\
& + 0.1591427(0.2500 - 0.38500f\pi^2)^2 + 0.0008360536(f\pi^2)^2 \\
& = 0.08328125(f\pi^2)^2 - 0.3125000(f\pi^2) + 0.5625000 \\
& - (0.0198928 - 0.0598560f\pi^2 + 0.0450006f\pi^4) - 0.001672122(f\pi^2)^2 \\
& + (0.00994642 - 0.03063497f\pi^2 + 0.02358693f\pi^4) + 0.0008360536(f\pi^2)^2 \\
& = 0.06103351(f\pi^2)^2 - 0.28327897(f\pi^2) + 0.5525362
\end{aligned}$$

$$6\pi^2 \frac{\tau}{E} \left(\frac{a}{R}\right)^2 = \left(\frac{a}{R}\right)^4 \left[0.24413404 \pi^4 f^2 - 0.84983691 \pi^2 f + 1.1050124 \right] \quad \underline{\underline{477}}$$

$$+ \left(\frac{f}{R}\right)^2 \frac{16\pi^4}{3}$$

$$K = f^2 \left[0.040689 \pi^2 f^2 - 0.1416395 f + \frac{0.1841787}{\pi^2} \right] + \frac{f}{9} \pi^2 \frac{1}{f^2}$$

now $\frac{f}{t} = \frac{R}{t} \frac{1}{2} \left(\frac{a}{R}\right)^2 f, \quad \text{or} \quad f = \frac{2 \left(\frac{f}{t}\right)}{f^2}$

$$\therefore K = \pi^2 \left[0.162756 \left(\frac{f}{t}\right)^2 + \frac{f}{9} \right] \frac{1}{f^2} - 0.2832790 \left(\frac{f}{t}\right) + \frac{0.1841787}{\pi^2} \frac{1}{f^2}$$

$$\text{Thus } K = 2 \left[0.0299762 \left(\frac{f}{t}\right)^2 + 0.1637144 \right] \frac{1}{f^2} - 0.2832790 \left(\frac{f}{t}\right)$$

$$\frac{0.0599524 \left(\frac{f}{t}\right)}{\left[0.0299762 \left(\frac{f}{t}\right)^2 + 0.1637144 \right]^{3/2}} = 0.2832790$$

$$0.00359429 \left(\frac{f}{t}\right)^2 = 0.00240550 \left(\frac{f}{t}\right)^2 + 0.01313759$$

$$\left(\frac{f}{t}\right)_{\min}^2 = \frac{0.01313759}{0.00118879} = 11.05123, \quad \left(\frac{f}{t}\right)_{\min} = 3.32434$$

$$K_{\min} = \left(\frac{0.1199048}{0.2832790} - 0.2832790 \right) 3.32434 = 0.4653930$$

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$$j = \pi \left[\frac{0.883685 \frac{1}{4}}{+ 4.626230} \right]^{\frac{1}{4}}$$

$$j_0 = \pi \sqrt[4]{4.826230} = \pi \sqrt{2.196868} = 1.482183 \pi$$

$$j_{min} = \pi \sqrt[4]{14.592036} = \pi \sqrt{3.819952} = 1.954469 \pi$$

①	②	③	④	⑤	⑥		
$\frac{f}{f_0}$	$0.2832790 \left(\frac{f}{f_0}\right)$	$0.0299762 \left(\frac{f}{f_0}\right)^2$	③ $+0.1637144$	④ $\frac{1}{2}$	K		
0	0	0	0.1637144	0.404617	0.809234		
1	0.2832790	0.0299762	0.1936906	0.440104	0.596929		
2	0.5665580	0.119948	0.2836191	0.532559	0.498560		
3	0.8498370	0.2697858	0.4335002	0.658407	0.461977		
4	1.1331160	0.4796192	0.6433336	0.802081	0.471046		
5	1.4163950	0.7494050	0.9131194	0.955573	0.494751		
6	1.6996740	1.0791732	1.242726	1.1148352	0.529996		
7	1.9829530	1.4668338	1.6325462	1.277713	0.572473		
8	2.2662320	1.9177262	2.082741	1.442979	0.617726		
9	2.5495110	2.4230722					
10	2.8327900	2.9972202					
11	3.1160690	3.6271202					

$$\lambda = \underline{\underline{1.0000}}$$

If the wave length in y -direction is (a/μ) instead of a

$$\nabla^4 F = \frac{E}{\rho^2} \frac{(1+\mu^2)}{2} \left[-\frac{1}{4} \frac{\pi^2}{a^2} \cos \frac{2\pi y}{a} + \left\{ 1 - \frac{1+\mu^2}{4} \right\} \cos \frac{2\pi y}{a} + \left\{ 1 - \frac{1+\mu^2}{2} \right\} \cos \frac{2\pi y}{a} \cos \frac{2\pi x}{a} - \frac{1}{4} \frac{\pi^2}{a^2} \cos \frac{4\pi x}{a} + \cos \frac{4\pi x}{a} \cos \frac{2\pi y}{a} + \cos \frac{2\pi x}{a} \cos \frac{4\pi y}{a} \right]$$

hence the particular integral is

$$E \left(\frac{a}{\rho} \right)^2 \frac{1+\mu^2}{8} \left[-\frac{1}{4} \frac{\pi^2}{a^2} \frac{\cos \frac{2\pi y}{a}}{(1+\mu^2)^2} + \frac{1}{4\mu^2} \left\{ 1 - \frac{1+\mu^2}{4} \right\} \cos \frac{2\pi y}{a} + \frac{1}{(1+\mu^2)^2} \left\{ 1 - \frac{1+\mu^2}{2} \right\} \cos \frac{2\pi x}{a} \cos \frac{2\pi y}{a} - \frac{1}{4} \frac{\pi^2}{a^2} \frac{\cos \frac{4\pi x}{a}}{(1+\mu^2)^2} + \frac{1}{4\mu^2} \frac{\cos \frac{4\pi x}{a}}{(1+\mu^2)^2} + \frac{1}{(4+\mu^2)^2} \frac{\cos \frac{4\pi x}{a} \cos \frac{2\pi y}{a}}{(2\pi)^2} + \frac{1}{(1+\mu^2)^2} \frac{\cos \frac{2\pi x}{a} \cos \frac{4\pi y}{a}}{(2\pi)^2} \right]$$

$$F = \frac{E}{(2\pi)^2} \left[A_2 \cosh \left(\frac{2\pi y}{a} \right) + B_2 \left(\frac{2\pi y}{a} \right) \sinh \left(\frac{2\pi y}{a} \right) \right] \cos \frac{2\pi x}{a} + \frac{E}{(4\pi)^2} \left[A_4 \cosh \left(\frac{4\pi y}{a} \right) + B_4 \left(\frac{4\pi y}{a} \right) \sinh \left(\frac{4\pi y}{a} \right) \right] \cos \frac{4\pi x}{a} + E a^2 \frac{x^2}{2} C$$

$$\begin{aligned}
 \frac{\partial \tilde{\gamma}}{\partial E} = & - \left[A_2 \cosh\left(\frac{2\pi\lambda}{a}\right) + B_2 \left(\frac{2\pi\lambda}{a}\right) \sinh\left(\frac{2\pi\lambda}{a}\right) \right] \cosh\left(\frac{2\pi\lambda}{a}\right) - \left[A_4 \cosh\left(\frac{4\pi\lambda}{a}\right) + B_4 \left(\frac{4\pi\lambda}{a}\right) \sinh\left(\frac{4\pi\lambda}{a}\right) \right] \cosh\left(\frac{4\pi\lambda}{a}\right) \\
 & + 2C + \left(\frac{a}{b}\right)^2 \frac{f^2}{g} \left[-\frac{f}{4} \frac{\pi^2}{\lambda^2} \cosh\left(\frac{2\pi\lambda}{a}\right) - \frac{1}{(1+\lambda)^2} \left(1 - \frac{\lambda^2}{2}\right) \cosh\left(\frac{2\pi\lambda}{a}\right) \cosh\left(\frac{2\pi\lambda}{a}\right) \right. \\
 & \left. + \frac{f\pi^2}{4} \left\{ \frac{1}{4} \cosh\left(\frac{4\pi\lambda}{a}\right) + \frac{4}{(4+\lambda)^2} \cosh\left(\frac{2\pi\lambda}{a}\right) \cosh\left(\frac{2\pi\lambda}{a}\right) + \frac{1}{(1+\lambda)^2} \cosh\left(\frac{2\pi\lambda}{a}\right) \cosh\left(\frac{4\pi\lambda}{a}\right) \right\} \right]
 \end{aligned}$$

$a_2 \cosh 2\pi + 2\pi b_2 \sinh 2\pi = \frac{f}{4} \frac{\pi^2}{\lambda^2} + \frac{1}{(1+\lambda)^2} \left(\frac{f\pi^2}{2} - 1 \right) + \frac{1}{4(1+\lambda)^2} \frac{f\pi^2}{\lambda^2}$
$(a_2 + b_2) \sinh 2\pi + 2\pi b_2 \cosh 2\pi = 0$
$a_4 \cosh 4\pi + 4\pi b_4 \sinh 4\pi = \frac{f\pi^2}{4} \left(\frac{f}{4} + \frac{4}{(4+\lambda)^2} \right)$
$(a_4 + b_4) \sinh 4\pi + 4\pi b_4 \cosh 4\pi = 0$

$$\begin{aligned}
\frac{1}{abE^2} \int_0^a \int_0^b \psi_4^2 dx dy &= \left(\frac{a}{R}\right)^4 \frac{(4b)^2}{256} \left[\frac{1}{8} (4\pi)^2 + \frac{1}{(1+\lambda)^4} \left(\frac{1}{2} \pi^2 - 1 \right)^2 + \frac{(4\pi)^2}{(4+\lambda^2)^4} + \frac{(4\pi)^2}{16(1+\lambda^2)^4} + \frac{(4\pi)^2}{128} \right] \\
&- \left(\frac{a}{R}\right)^4 \frac{(4b)^2}{64} \left[\frac{1}{4} \pi^2 \left\{ \frac{(a_2 - b_2)}{9\pi} \sinh 2\pi + b_2 \cosh 2\pi \right\} + \frac{1}{(4\pi)^2} \left(\frac{1}{2} \pi^2 - 1 \right) \left\{ a_2 \frac{\sinh 2\pi}{4\pi} + b_2 \frac{\cosh 2\pi}{2} \right\} \right. \\
&+ \left. \frac{1}{4(1+\lambda^2)^2} \left\{ \frac{1}{10\pi} a_2 \sinh 8\pi + b_2 \left(\frac{1}{5} \cosh 2\pi + \frac{3}{50\pi} \sinh 2\pi \right) \right\} \right. \\
&+ \left. \frac{1}{16} \pi^2 \left\{ \frac{(a_4 - b_4)}{4\pi} \sinh 4\pi + b_4 \cosh 4\pi \right\} + \frac{1}{(1+\lambda^2)^2} \left\{ a_4 \frac{\sinh 4\pi}{5\pi} + b_4 \left(\frac{4}{5} \cosh 4\pi - \frac{5}{25\pi} \sinh 4\pi \right) \right\} \right] \\
&+ \left(\frac{a}{R}\right)^4 \frac{(4b)^2}{128} \left[\frac{a_2^2}{2} \left(\frac{\sinh 4\pi}{4\pi} + 1 \right) + \frac{a_2 b_2}{2} \left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi} \right) + \frac{b_2^2}{2} \left(\frac{16\pi^2 + 2}{16\pi} \sinh 4\pi - \frac{1}{2} \cosh 4\pi - \frac{4\pi^2}{3} \right) \right. \\
&+ \left. \frac{a_4^2}{2} \left(\frac{\sinh 8\pi}{8\pi} + 1 \right) + \frac{a_4 b_4}{2} \left(\cosh 8\pi - \frac{\sinh 8\pi}{4\pi} \right) + \frac{b_4^2}{2} \left(\frac{64\pi^2 + 2}{32\pi} \sinh 8\pi - \frac{1}{2} \cosh 8\pi - \frac{16\pi^2}{3} \right) \right] \\
&= \mathcal{A}_1
\end{aligned}$$

$$\frac{1}{abE^2} \int_0^a \int_0^b \phi_1^2 dx dy = \lambda^4 \left(\frac{a}{R}\right)^4 \frac{(1+\lambda^2)^2}{32b} \left[\frac{2}{\lambda^8} \left(\frac{1+\pi^2}{4} - 1\right)^2 + \frac{1}{(1+\lambda^2)^4} \left(\frac{1+\pi^2}{2} - 1\right)^2 + \frac{2}{\lambda^8} \left(\frac{1+\pi^2}{4}\right)^2 + \frac{(1+\pi^2)^2}{16(1+\lambda^2)^4} \right]$$

$$+ \frac{(1+\pi^2)^2}{(1+4\lambda^2)^4} \Big]$$

$$+ \lambda^4 \left(\frac{a}{R}\right)^4 \frac{(1+\lambda^2)^2}{64} \left[\frac{1}{\lambda^4} \left(\frac{1+\pi^2}{4} - 1\right) \left\{ \frac{(a_2+b_2)}{2\pi} \sinh \frac{2\pi}{\lambda} + b_2 \cosh 2\pi \right\} + \frac{1}{(1+\lambda^2)^2} \left(\frac{1+\pi^2}{2} - 1\right) \left(\frac{a_2+b_2}{4\pi} \frac{\sinh 2\pi}{\lambda} + b_2 \frac{\cosh 2\pi}{2} \right) \right]$$

$$+ \frac{1}{(1+4\lambda^2)^2} \left\{ \frac{1}{10\pi} (a_2+2b_2) \sinh 2\pi + b_2 \left(\frac{1}{5} \cosh 2\pi + \frac{3}{50\pi} \sinh 2\pi \right) \right\}$$

$$+ \frac{1+\pi^2}{16\lambda^4} \left\{ \frac{(a_4+b_4)}{4\pi} \sinh 4\pi + b_4 \sinh 4\pi \right\} + \frac{1}{4(1+\lambda^2)^2} \left\{ \frac{1}{5\pi} (a_4+2b_4) \sinh 4\pi + b_4 \left(\frac{4}{5} \cosh 4\pi - \frac{3}{25\pi} \sinh 4\pi \right) \right\}$$

$$+ \left(\frac{a}{R}\right)^4 \lambda^4 \frac{(1+\lambda^2)^2}{128} \left[\frac{(a_2+2b_2)^2}{2} \left(\frac{\sinh 4\pi}{4\pi} + 1 \right) + \frac{(a_2+2b_2)b_2}{2} \left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi} \right) + \frac{b_2^2}{2} \left(\frac{16\pi^2+2}{16\pi} \sinh 4\pi - \frac{1}{2} \cosh 4\pi - \frac{6\pi}{3} \right) \right]$$

$$+ \frac{(a_4+2b_4)^2}{2} \left(\frac{\sinh 8\pi}{8\pi} + 1 \right) + \frac{(a_4+2b_4)b_4}{2} \left(\cosh 8\pi - \frac{\sinh 8\pi}{8\pi} \right) + \frac{b_4^2}{2} \left(\frac{64\pi^2+2}{32\pi} \sinh 8\pi - \frac{1}{2} \cosh 8\pi - \frac{16\pi}{3} \right) \Big]$$

$$= A_2$$

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$$\begin{aligned}
\frac{1}{abE^2} \int_0^1 \int_0^1 x y \, dx dy &= b^2 \left(\frac{a}{R} \right)^4 \frac{L^2 \lambda^2 \pi^2}{32} \left[\frac{1}{(1+\lambda^2)^4} \left(\frac{\lambda^2}{2} - 1 \right)^2 + \frac{(\lambda^2)^2}{4(4+\lambda^2)^4} + \frac{(4\lambda^2)^2}{4(1+4\lambda^2)^2} \right] \\
&+ b^2 \left(\frac{a}{R} \right)^4 \frac{L^2 \lambda^2}{64} \left[\frac{1}{(1+\lambda^2)^2} \left(\frac{-\lambda^2}{2} - 1 \right) \left\{ - (a_2 + b_2) \frac{\sinh 2\pi}{4\pi} + b_2 \left(\frac{\sinh 2\pi}{4\pi} - \frac{1}{2} \cosh 2\pi \right) \right\} \right. \\
&+ \frac{1}{2(1+4\lambda^2)^2} \left\{ - (a_2 + b_2) \frac{\sinh 2\pi}{5\pi} + b_2 \left(-\frac{2}{5} \cosh 2\pi + \frac{2}{25\pi} \sinh 2\pi \right) \right\} \\
&+ \frac{1}{2(4+\lambda^2)^2} \left\{ - (a_4 + b_4) \frac{\sinh 4\pi}{12\pi} + b_4 \left(-\frac{2}{5} \cosh 4\pi + \frac{4}{25\pi} \sinh 4\pi \right) \right\} \Bigg] \\
&+ b^2 \left(\frac{a}{R} \right)^4 \frac{L^2 \lambda^2}{128} \left[\frac{(a_2 + b_2)^2}{2} \left\{ \frac{\sinh 4\pi}{4\pi} - 1 \right\} + \frac{b_2(a_2 + b_2)}{2} \left\{ \cosh 4\pi - \frac{\sinh 4\pi}{4\pi} \right\} + \frac{b_2^2}{2} \left\{ \frac{4\pi^2}{3} + \frac{16\pi^2 + 2}{16\pi} \sinh 4\pi - \frac{\cosh 4\pi}{2} \right\} \right. \\
&+ \frac{(a_4 + b_4)^2}{2} \left\{ \frac{\sinh 4\pi}{8\pi} - 1 \right\} + \frac{b_4(a_4 + b_4)}{2} \left\{ \cosh 8\pi - \frac{\sinh 4\pi}{8\pi} \right\} + \frac{b_4^2}{2} \left\{ \frac{16\pi^2}{3} + \frac{64\pi^2 + 2}{32\pi} \sinh 8\pi - \frac{\cosh 8\pi}{2} \right\} \Bigg]
\end{aligned}$$

A_3

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$$\begin{aligned} \frac{1}{a} \int_0^a \sin \frac{2\pi x}{a} \sinh(\sqrt{2}\lambda) \frac{\pi x}{a} dx &= \frac{1}{2\pi i} \int_0^a \left[\cosh \frac{\pi x}{a} (\sqrt{2}\lambda + 2i) - \cosh \frac{\pi x}{a} (\sqrt{2}\lambda - 2i) \right] d\left(\frac{\pi x}{a}\right) \\ &= \frac{1}{2\pi i} \left[\frac{1}{\sqrt{2}\lambda + 2i} \sinh \pi (\sqrt{2}\lambda + 2i) - \frac{1}{\sqrt{2}\lambda - 2i} \sinh \pi (\sqrt{2}\lambda - 2i) \right] = -\frac{1}{\pi} \frac{\sinh \pi \sqrt{2}\lambda}{4 + 2\lambda^2} \end{aligned}$$

$$\frac{1}{a} \int_0^a \frac{\pi x}{a} \sin \frac{2\pi x}{a} dx = \frac{1}{4\pi} \left[\sinh \pi - \theta \cosh \theta \right]_0^{2\pi} = -\frac{1}{2}$$

$$\begin{aligned} \frac{1}{a} \int_0^a \frac{\pi x}{a} \sin \frac{2\pi x}{a} \cos \frac{\pi x}{a} dx &= \frac{1}{2\pi} \int_0^a \frac{\pi x}{a} \left[\sin \frac{3\pi x}{a} + \sin \frac{\pi x}{a} \right] d\left(\frac{\pi x}{a}\right) \\ &= \frac{1}{2\pi} \left[\frac{1}{9} (\sinh \theta - \theta \cosh \theta)_0^{3\pi} + (\sinh \theta - \theta \cosh \theta)_0^{\pi} \right] = \frac{1}{2} \left[\frac{1}{3} + 1 \right] = \frac{2}{3} \end{aligned}$$

$$\frac{1}{a} \int_0^a \sinh^2 \frac{\sqrt{2}\lambda \pi x}{a} dx = \frac{1}{2\pi} \int_0^a (\cosh 2\sqrt{2}\lambda \theta - 1) d\theta = \frac{1}{2\pi} \left[\frac{\sinh 2\sqrt{2}\lambda \pi}{2\sqrt{2}\lambda} - \pi \right] = \frac{\sinh 2\sqrt{2}\lambda \pi}{4\sqrt{2}\lambda \pi} - \frac{1}{2}$$

$$\frac{1}{a} \int_0^a \frac{\pi x}{a} \sinh(\sqrt{2}\lambda) \frac{\pi x}{a} dx = \frac{1}{2\lambda^2 \pi} \left[-\sqrt{2}\lambda \pi \cosh \sqrt{2}\lambda \pi - \sinh \sqrt{2}\lambda \pi \right]$$

$$\begin{aligned} \frac{1}{a} \int_0^a \left(\frac{\pi x}{a}\right) \cos \frac{\pi x}{a} \sinh \sqrt{2}\lambda \left(\frac{\pi x}{a}\right) dx &= \frac{1}{2\pi} \int_0^{\pi} \theta \left[\sinh \theta (\sqrt{2}\lambda + i) + \sinh \theta (\sqrt{2}\lambda - i) \right] d\theta \\ &= \frac{1}{2\pi} \left[\frac{1}{(\sqrt{2}\lambda + i)^2} \left\{ (\sqrt{2}\lambda + i) \pi \cosh \pi (\sqrt{2}\lambda + i) - \sinh \pi (\sqrt{2}\lambda + i) \right\} \right. \\ &\quad \left. + \frac{1}{(\sqrt{2}\lambda - i)^2} \left\{ (\sqrt{2}\lambda - i) \pi \cosh \pi (\sqrt{2}\lambda - i) - \sinh \pi (\sqrt{2}\lambda - i) \right\} \right] \\ &= -\frac{\sqrt{2}\lambda \cosh \sqrt{2}\lambda \pi}{1 + 2\lambda^2} + \frac{1}{\pi} \frac{(2\lambda^2 - 1) \sinh \sqrt{2}\lambda \pi}{(1 + 2\lambda^2)^2} \end{aligned}$$

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$$\frac{1}{a} \int_0^{\frac{\pi}{2}} \left(\frac{\pi x}{a} \right)^2 dx = \pi^2 \int_0^1 \theta^2 d\theta = \frac{\pi^2}{3}$$

$$\frac{1}{4} \int_0^{\frac{\pi}{2}} \left(\frac{\pi x}{a} \right)^2 \cos \frac{\pi x}{a} dx = \frac{1}{\pi} \int_0^{\pi} \theta^2 \cos \theta d\theta = \frac{1}{\pi} \left[-2\pi \right] = -2$$

$$\frac{1}{a} \int_0^{\frac{\pi}{2}} \left(\frac{\pi x}{a} \right)^2 \cos \frac{\pi x}{a} dx = \frac{1}{\pi} \int_0^{\pi} \theta^2 \cos \theta d\theta = \frac{1}{\pi} \int_0^{\pi} \theta^2 (1 + \cos 2\theta) d\theta = \frac{1}{\pi} \left[\frac{\pi^3}{3} + \frac{1}{8} (4\pi) \right] = \frac{\pi^2}{6} + \frac{1}{4}$$

$$\frac{1}{E^2 ab} \int_0^a \int_0^b x^2 y^2 dx dy = \left(\frac{1}{b} \right)^4 \left(\frac{1}{8} \right)^2 \left[\frac{1}{4} \left\{ \frac{\pi^2}{8} \frac{a^2}{(1+2\lambda)^2} + \frac{1+\lambda^2}{(1+2\lambda)^2} \right\} + \frac{1}{4} \left(\frac{\pi^2}{32} \right) \left(\frac{a^2}{2+\lambda^2} \right)^2 + \frac{1}{4} \left(\frac{\pi^2}{16} \right)^2 \left(\frac{3\lambda^2}{1+2\lambda^2} \right)^2 \right]$$

$$+ \frac{1}{4} \left(\frac{\pi^2}{16} \right)^2 \left(\frac{3\lambda^2}{1+2\lambda^2} \right)^2 + \frac{1}{4} \left(\frac{\pi^2}{16} \right)^2 \left(\frac{3\lambda^2}{1+2\lambda^2} \right)^2 + \frac{1}{4} \left(\frac{\pi^2}{16} \right)^2 \left(\frac{3\lambda^2}{1+2\lambda^2} \right)^2$$

$$+ \left\{ \frac{\pi^2}{8} \frac{\lambda^2}{(1+2\lambda^2)} + \frac{1+\lambda^2}{(1+2\lambda^2)^2} \right\} \left\{ \frac{1}{2} \frac{1}{(1+2\lambda^2)^2} - \frac{1}{2} + \frac{1}{4} \left(\frac{\lambda^2}{1+2\lambda^2} \right) \right\}$$

$$+ \frac{\pi^2}{32} \frac{\lambda^2}{(2+\lambda^2)} \left\{ -\frac{1}{2} \frac{1}{(1+2\lambda^2)(2+\lambda^2)} + \frac{1}{4} - \frac{2}{3} \frac{\lambda^2}{(1+2\lambda^2)} \right\}$$

$$+ \frac{1}{2} \left\{ \frac{(\sinh 2\sqrt{2}\lambda\pi)}{4\sqrt{2}\lambda\pi} - \frac{1}{2} \right\} \pi^2 - \frac{1}{2} \frac{\pi}{(1+2\lambda^2)^2 (\sinh \sqrt{2}\lambda\pi)} - \frac{1}{2} \frac{\pi}{(1+2\lambda^2)^2 (\sinh \sqrt{2}\lambda\pi)} \frac{1}{2\lambda^2} \left[\sqrt{2}\lambda\pi \coth \sqrt{2}\lambda\pi - \sinh \sqrt{2}\lambda\pi \right]$$

$$- \frac{\lambda^2}{(1+2\lambda^2)} \frac{\pi}{(1+2\lambda^2) \sinh(\sqrt{2}\lambda\pi)} \left[-\frac{\sqrt{2}\lambda\pi \coth \sqrt{2}\lambda\pi}{1+2\lambda^2} + \frac{1}{\pi} \frac{(2\lambda^2-1) \sinh \sqrt{2}\lambda\pi}{(1+2\lambda^2)^2} \right] + \frac{\pi^2}{16}$$

$$- \frac{2\lambda^2}{(1+2\lambda^2)^2} + \frac{\lambda^4}{(1+2\lambda^2)^2} \left(\frac{\pi^2}{6} + \frac{1}{4} \right) \left. \right\}$$

$$\frac{E_1}{2Eab^3} = \left(\frac{q}{8}\right)^2 \left[H_1(x)(-12)^2 + H_2(x)(-12) + H_3(x) \right] + \left(\frac{q}{8}\right)^2 \left[\frac{1}{24} - \frac{3}{128}(12)^2 \right]$$

$$H_1(x) = \frac{105}{32768} + \frac{9x^4}{32768} + \frac{77x^4}{65536} + \frac{(1-4x^2)x^4}{2048(1+8x^2)^2} + \frac{(1-x^2)x^4}{32768(1+2x^2)^2} + \frac{x^4}{512(1+2x^2)^2}$$

$$+ \frac{x^4}{2048(2+x^2)^2} + \frac{x^6}{256(1+2x^2)^2} + \frac{x^6}{4096(2+x^2)^2} + \frac{9x^6}{1024(1+8x^2)^2} + \frac{9x^6}{85536(1+2x^2)^2}$$

$$+ \frac{9x^6}{65536(1+2x^2)^2} + \frac{9x^6}{1024(1+8x^2)^2} + \frac{9x^6}{85536(1+2x^2)^2}$$

$$H_1(x) = \frac{105}{32768} + \frac{95x^4}{65536} + \frac{x^4(1+2x^2)}{2048(1+8x^2)^2} + \frac{x^4(120+x^2)}{65536(1+2x^2)^2} + \frac{3x^6}{4096(2+x^2)^2}$$

$$\begin{aligned}
 H_2(\lambda) = & -\frac{5}{48} - \frac{3}{256} \lambda^2 - \frac{\lambda^4}{32(1+\lambda)^2} - \frac{\lambda^4}{26(1+\lambda)^2} - \frac{\lambda^4}{16(1+2\lambda)^2} - \frac{\lambda^4}{96(2+\lambda)^2(1+\lambda)^2} + \frac{\lambda^4}{64(1+2\lambda)^2(2+\lambda)^2} \\
 & + \frac{\lambda^4(1+\lambda^2)}{16(1+2\lambda)^2} + \frac{\lambda^4}{16(1+2\lambda)^2} - \frac{\lambda^4}{16(1+2\lambda)^2} + \frac{\lambda^4}{32(1+2\lambda)^2} - \frac{\lambda^4}{64(1+2\lambda)^2(2+\lambda)^2} + \frac{\lambda^4}{128(2+\lambda)^2} \\
 & - \frac{\lambda^6}{48(1+2\lambda)^2(2+\lambda)^2}
 \end{aligned}$$

$$H_2(b) = -\frac{5}{48} - \frac{3\lambda^2}{256} - \frac{3\lambda^4}{64(1+2\lambda)^2} - \frac{\lambda^4}{384(2+\lambda)^2}$$

$$\begin{aligned}
 H_3(\lambda) = & \frac{\pi^2}{8} - \frac{3}{16} + \frac{\lambda^4}{8(1+2\lambda)^2} + \frac{\lambda^4}{16} + \frac{\lambda^4}{8(1+2\lambda)^2} - \frac{\lambda^4}{4(1+2\lambda)^2} - \frac{1}{8(1+2\lambda)^2} \\
 & + \frac{\lambda^4}{4(1+2\lambda)^2} \left(\frac{\pi^2}{6} - \frac{1}{4} \right) + \frac{\lambda^2}{4(1+2\lambda)^2} (\sqrt{2}\lambda\pi) \operatorname{arctanh} \sqrt{2}\lambda\pi - \frac{\lambda^4}{(1+2\lambda)^4} + \frac{\pi^2 \lambda^2 \left(\frac{1}{2} + \frac{1}{4\sqrt{2}\pi} \operatorname{arctanh} 2\sqrt{2}\lambda\pi \right)}{4(1+2\lambda)^2 (\cosh 2\sqrt{2}\lambda\pi - 1)} \\
 & + \frac{\lambda^2(1+\lambda)^2}{4(1+2\lambda)^4} + \frac{\lambda^2(1+\lambda^2)}{2(1+2\lambda)^4} - \frac{\lambda^2(1+\lambda^2)}{2(1+2\lambda)^2} + \frac{\lambda^4(1+\lambda^2)}{4(1+2\lambda)^2} + \frac{\pi^2 \lambda^2 \left(-\frac{1}{2} + \frac{1}{4\sqrt{2}\pi} \operatorname{arctanh} \sqrt{2}\lambda\pi \right)}{4(1+2\lambda)^2 (\cosh 2\sqrt{2}\lambda\pi - 1)} \\
 & - \frac{1}{8(1+2\lambda)^2} (\sqrt{2}\lambda\pi) \operatorname{arctanh} \sqrt{2}\lambda\pi + \frac{1}{8(1+2\lambda)^2} + \frac{\lambda^4}{2(1+2\lambda)^2} \sqrt{2}\lambda\pi \operatorname{arctanh} \sqrt{2}\lambda\pi + \frac{(1-2\lambda)^2 \lambda^4}{2(1+2\lambda)^4} + \frac{\lambda^2}{2\lambda} \lambda^2 \\
 & - \frac{\lambda^4}{(1+2\lambda)^2} + \frac{\lambda^6}{2(1+2\lambda)^2} \left(\frac{\pi^2}{6} + \frac{1}{4} \right)
 \end{aligned}$$

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$$H_3(\lambda) = \frac{\pi^2}{8} + \frac{3\lambda^2(2-\lambda^2)}{8(1+\lambda^2)^3} + \frac{\lambda^4(3+2\lambda^2)}{8(1+\lambda^2)^4} - \frac{\lambda^2(1+4\lambda^2)}{4(1+\lambda^2)^2} - \frac{\lambda^2(8+9\lambda^2-2\lambda^4)}{16(1+\lambda^2)^2} \\ + \frac{\pi^2 \lambda^2}{24} \left[\frac{\lambda^2}{(1+\lambda^2)} + 1 \right] + \left[\frac{\lambda^2}{4(1+\lambda^2)^2} - \frac{1}{8(1+\lambda^2)} + \frac{1}{16(1+\lambda^2)^2} \right] \sqrt{2} \pi \lambda \operatorname{ci}(\lambda \sqrt{2} \pi)$$

$$H_3(\lambda) = \frac{\pi^2}{8} + \frac{\lambda^2(3+\lambda^4)}{4(1+\lambda^2)^3} - \frac{3\lambda^2(4+11\lambda^2+10\lambda^4)}{16(1+\lambda^2)^2} + \frac{\pi^2 \lambda^2}{24} \left[\frac{\lambda^2}{(1+\lambda^2)} + 1 \right]$$

$$- \frac{1}{16(1+\lambda^2)^2} \sqrt{2} \pi \lambda \operatorname{ci}(\lambda \sqrt{2} \pi)$$

$$\frac{E_2}{E_{at t}} = \frac{1}{12} \left(\frac{1}{R} \right)^2 \left(\frac{H\pi^2}{8} \right)^2 \left[\frac{1}{8} + \frac{3}{8} \pi^4 + \frac{1}{4} \pi^2 \right] = \left(\frac{1}{R} \right)^2 \left(\frac{H\pi^2}{8} \right)^2 \left[\frac{1}{32} (1 + \pi^4) + \frac{1}{16} \pi^2 \right]$$

Thus total energy expression

$$= \left(\frac{Q}{R} \right)^4 \left(\frac{1}{8} \right)^2 \left[H_1(\lambda) H(\pi^2) + H_2(\lambda) H(\pi^2) + H_3(\lambda) \right] + \left(\frac{1}{R} \right)^2 \left(\frac{1}{8} \right)^2 \left[\frac{1}{32} (1 + \pi^4) + \frac{1}{16} \pi^2 \right] \pi^4$$

$$- \left(\frac{Q}{R} \right)^2 \left(\frac{1}{8} \right)^2 \pi^2 \left(\frac{Q}{8} \right)^2 \frac{Q}{E} - \frac{1}{9} \left(\frac{Q}{E} \right)^2$$

$$\therefore \frac{Q}{E} 3\pi^2 \frac{\pi^2}{4} = \left(\frac{Q}{R} \right)^2 \left[4H_1 + \pi^4 + 3H_2 + \pi^2 + 2H_3 \right] + \frac{\left(\frac{1}{R} \right)^2}{\left(\frac{1}{8} \right)^2} \left[\frac{1}{16} (1 + \pi^4) + \frac{1}{24} \pi^2 \right] \pi^4$$

$$\pi^2 H = \pi^2 \left[\frac{16}{3} H_1 + \pi^2 + 4H_2 + 1 + \frac{8H_3}{3\pi^2} \right] + \frac{\pi^2}{\pi^2} \left[\frac{1}{12} (1 + \pi^4) + \frac{1}{18} \pi^2 \right]$$

$$= \frac{\pi^2}{\pi^2} \left[\frac{64}{3} H_1 \left(\frac{1}{\pi^2} \right)^2 + \left\{ \frac{1}{12} (1 + \pi^4) + \frac{1}{18} \pi^2 \right\} + \frac{8H_3}{3} \frac{\pi^2}{\pi^2} + 8H_2 \left(\frac{1}{\pi^2} \right) \right]$$

$$= 2 \left[\frac{512}{9} H_1 H_3 \left(\frac{1}{\pi^2} \right)^2 + H_3 \left\{ \frac{2}{9} (1 + \pi^4) + \frac{4}{27} \pi^2 \right\} + 8H_2 \left(\frac{1}{\pi^2} \right) \right]$$

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X

$$\begin{aligned}
& -a_2 \frac{\sinh 2\pi}{2\pi} \left\{ \left[\frac{1}{4(1+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2} + \frac{1}{20(1+4\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)^2} \right] (H\pi') + \left[1 - \frac{1}{2(1+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2} \right] \right\} \\
& + b_2 \left[\frac{\sinh 2\pi}{2\pi} \left\{ \left[\frac{1}{2} + \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{\lambda^2(\lambda^2-1)}{5(1+4\lambda^2)^2} + \frac{(2\lambda^2+3)(\lambda^2-1)}{100(1+4\lambda^2)^2} \right] (H\pi') - \left[1 + \frac{\lambda^4}{(1+\lambda^2)^2} \right] \right\} \right. \\
& \quad \left. - \cosh 2\pi \left\{ \left[\frac{1}{4(1+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2} + \frac{1}{20(1+4\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)^2} \right] (H\pi') + \left[1 - \frac{1}{2(1+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2} \right] \right\} \right] \\
& - a_4 \frac{\sinh 4\pi}{4\pi} (H\pi') \left\{ \frac{1}{5(4+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2} \right\} \\
& + b_4 (H\pi') \left[\frac{\sinh 4\pi}{4\pi} \left\{ \frac{1}{8} + \frac{(2+3\lambda^2)(6-\lambda^2)}{25(4+\lambda^2)^2} - \frac{2\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2} \right\} - \cosh 4\pi \left\{ \frac{1}{5(4+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2} \right\} \right]
\end{aligned}$$

$$\cosh 2\pi a_2 + 2\pi \sinh 2\pi b_2 = (H\pi') \left[\frac{1}{4} + \frac{1}{2(1+\lambda^2)} + \frac{1}{4(1+4\lambda^2)^2} \right] - \frac{1}{(1+\lambda^2)^2}$$

$$\sinh 2\pi a_2 + (\sinh 2\pi + 2\pi \cosh 2\pi) b_2 = 0$$

$$\begin{aligned}
& \frac{1}{2} (\sinh 4\pi + 4\pi) b_2 = - \sinh 2\pi \left[\right. \\
& \quad \left. b_2 = - \frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi} + 1} \left[(H\pi') \left\{ \frac{1}{4} + \frac{1}{2(1+\lambda^2)} + \frac{1}{4(1+4\lambda^2)^2} \right\} - \frac{1}{(1+\lambda^2)^2} \right] \right. \\
& \quad \left. a_2 = + \frac{\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi}{\frac{\sinh 4\pi}{4\pi} + 1} \left[\right. \right]
\end{aligned}$$

$$a_4 = + \frac{\frac{\sinh 6\pi}{4\pi} + \cosh 6\pi}{\frac{\sinh 6\pi}{8\pi} + i} \quad \int \pi^2 \left\{ \frac{1}{16} + \frac{1}{(4+x)^2} \right\}$$

$$b_4 = - \frac{\frac{\sinh 6\pi}{4\pi}}{\frac{\sinh 6\pi}{8\pi} + i} \quad \int \pi^2 \left\{ \frac{1}{16} + \frac{1}{(4+x)^2} \right\}$$

$$\frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi} + 1} = \frac{42.613218}{11410.473} = 0.0037345707; \quad \frac{\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi}{\frac{\sinh 4\pi}{4\pi} + 1} = 0.02499735$$

$$\frac{\frac{\sinh 6\pi}{6\pi}}{\frac{\sinh 8\pi}{8\pi} + 1} = 0.000061746855, \quad \frac{\frac{\sinh 6\pi}{6\pi} + \cosh 6\pi}{\frac{\sinh 8\pi}{8\pi} + 1} = 0.00096621164$$

Let us first

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$$g_1(\lambda) = \frac{1}{4(1+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2} + \frac{1}{20(1+4\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)^2}$$

$$g_2(\lambda) = 1 - \frac{1}{2(1+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2}$$

$$g_3(\lambda) = \frac{1}{2} + \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{2}{5} \frac{\lambda^2(1-\lambda^2)}{(1+4\lambda^2)^2} + \frac{(2\lambda^2+3)(6\lambda^2-1)}{100(1+4\lambda^2)^2}$$

$$g_4(\lambda) = 1 + \frac{\lambda^4}{(1+\lambda^2)^2}$$

$$g_5(\lambda) = \frac{1}{5(4+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2}$$

$$g_6(\lambda) = \frac{1}{8} + \frac{(2+3\lambda^2)(6-\lambda^2)}{25(4+\lambda^2)^2} - \frac{2\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2}$$

$$h_1(\lambda) = \frac{1}{4} + \frac{1}{2(1+\lambda^2)^2} + \frac{1}{4(1-4\lambda^2)^2}$$

$$h_2(\lambda) = \frac{1}{(1+\lambda^2)^2}$$

$$h_3(\lambda) = \frac{1}{16} + \frac{1}{(4+\lambda^2)^2}$$

$$- \left[H_1(H_2^2) - H_2 \right] \left[\frac{\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi}{\frac{\sinh 4\pi}{4\pi} + 1} \cdot \frac{\sinh 2\pi}{2\pi} (g_1 + H_2^2 + g_2) + \frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi} + 1} - \frac{\sinh 2\pi}{2\pi} (g_3 - H_2^2 - g_4) \right]$$

$$- \frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi} + 1} \cosh 2\pi (g_1 + H_2^2 + g_2)$$

$$- H_3(H_2^2) \left[\frac{\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi}{\frac{\sinh 8\pi}{8\pi} + 1} - \frac{\sinh 4\pi}{4\pi} g_5 + \frac{\frac{\sinh 4\pi}{4\pi}}{\frac{\sinh 8\pi}{8\pi} + 1} - \frac{\sinh 4\pi}{4\pi} g_6 - \frac{\frac{\sinh 4\pi}{4\pi} \cdot \cosh 4\pi}{\frac{\sinh 8\pi}{8\pi} + 1} g_5 \right]$$

$$= - \frac{\left(\frac{\sinh 2\pi}{2\pi} \right)^2}{\frac{\sinh 4\pi}{4\pi} + 1} \nearrow \frac{0.15914286}{[H_1(H_2^2) - H_2]} \left[\underbrace{(g_1 + g_3)(H_2^2)}_{\frac{27}{100}} - \underbrace{(g_4 - g_2)}_{\frac{1}{2}} \right]$$

$$0.12254000 H_2^2$$

$$- 0.07957143$$

$$- \frac{\left(\frac{\sinh 4\pi}{4\pi} \right)^2}{\frac{\sinh 8\pi}{8\pi} + 1} \nearrow \frac{0.079571436}{H_3(H_2^2)^2} \left[\underbrace{g_5 + g_6}_{\frac{41}{200}} \right] = F_1$$

$$0.016313384$$

$$\begin{aligned}
F_2 = & \left[H_1(A^2) - H_2 \right]^2 \left[\left\{ (1+\lambda)^2 21102500 + 0.00018495(1-\lambda)^2 \right\} - \left\{ 0.2892467(3\lambda^4 + 2\lambda^2 - 1) \right. \right. \\
& + 3.64100103(1+\lambda)^2 - 0.00010158\lambda^2(1-\lambda)^2 \left. \right\} + \left\{ 0.03928203 \left[4\pi^2(1+\lambda)^2 + \frac{1}{2}(5\lambda^4 + 2\lambda^2 + 1) \right] + 0.24995787(3\lambda^4 + 2\lambda^2 - 1) \right. \\
& \left. \left. + 0.00003488 \left[2(2\lambda^2 - 1) - \frac{4\pi^2}{3}(1-\lambda)^2 \right] \right\} \right] \\
& + \left[H_2(A^2) \right]^2 \left[3.66148735(1+\lambda)^2 - \left\{ 0.26989437(3\lambda^4 + 2\lambda^2 - 1) + 6.78318640(1+\lambda)^2 \right\} \right. \\
& \left. + \left\{ 0.01989437 \left[16\pi^2(1+\lambda)^2 + \frac{1}{9}(5\lambda^4 + 2\lambda^2 + 1) \right] + 0.250000(3\lambda^4 + 2\lambda^2 - 1) \right\} \right]
\end{aligned}$$

$$\begin{aligned}
F_3 = & \left\{ \frac{17}{512} \left(1 + \frac{1}{\lambda^4} \right) + \frac{1}{16(1+\lambda^2)^2} + \frac{1}{64(4+\lambda^2)^2} + \frac{1}{64(1+4\lambda^2)^2} \right\} 6\pi^2 \lambda^2 \\
& - \left\{ \frac{1}{4\lambda^4} + \frac{1}{4(1+\lambda^2)^2} \right\} + \left\{ \frac{1}{2\lambda^4} + \frac{1}{4(1+\lambda^2)^2} \right\}
\end{aligned}$$

$$\lambda = 1.000$$

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$$\begin{aligned} F_3 &= (0.06640625 + 0.015625 + 0.00125)(f\pi^2)^2 - (0.25 + 0.0625)(f\pi^2) \\ &\quad + (0.5 + 0.0625) \\ &= 0.0832125(f\pi^2)^2 - 0.3125(f\pi^2) + 0.5625 \end{aligned}$$

$$H_1(\lambda) = 0.25 + 0.125 + 0.01 = 0.385, \quad H_2(\lambda) = 0.25.$$

$$H_3(\lambda) = 0.0625 + 0.04 = 0.1025$$

$$\begin{aligned} F_1 &= -[0.385(f\pi^2) - 0.25][0.12254000(f\pi^2) - 0.07957143] - 0.016313384 \times 0.1025(f\pi^2)^2 \\ &= -0.04885002(f\pi^2)^2 + 0.06127000(f\pi^2) - 0.01989286 \end{aligned}$$

$$\begin{aligned} F_2 &= [0.385(f\pi^2) - 0.25]^2 0.15920493 + (0.1025 f\pi^2)^2 0.07957575 \\ &= 0.02359815(f\pi^2)^2 - 0.03064695(f\pi^2) + 0.00995031 + 0.00083604(f\pi^2)^2 \end{aligned}$$

$$\begin{aligned} 6\pi^2 \frac{\sigma}{E} &= \left(\frac{g}{R}\right)^2 \left\{ 0.23546168 \pi^4 f^2 - 0.84563085 \pi^2 f + 0.55255745 \right\} \\ &\quad + \frac{\left(\frac{f}{R}\right)^2}{\left(\frac{g}{R}\right)^2} \pi^2 \frac{16}{3} \end{aligned}$$

$$\begin{aligned} K &= g^2 \left\{ 0.03924361 \pi^2 f^2 - 0.14093848 f + \frac{0.09209291}{\pi^2} \right\} + \frac{1}{f^2} \pi^2 0.8888889 \\ &= \frac{\pi^2}{f^2} \left\{ 0.15692444 \left(\frac{f}{E}\right)^2 + 0.88888889 \right\} + \frac{0.09209291}{\pi^2} \frac{\pi^2}{f^2} - 0.28187696 \left(\frac{f}{E}\right) \end{aligned}$$

$$K = 2 \left\{ 0.02891247 \left(\frac{f}{E} \right)^2 + 0.16372073 \right\}^{\frac{1}{2}} - 0.28187696 \left(\frac{f}{E} \right)$$

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$$K_0 = 0.8092$$

$$0.05782494 \left(\frac{f}{E} \right) = 0.28187696 \left\{ 0.02891247 \left(\frac{f}{E} \right)^2 + 0.16372073 \right\}^{\frac{1}{2}}$$

$$0.00334372 \left(\frac{f}{E} \right)^2 = 0.01300837,$$

$$- 0.00229723$$

$$\left(\frac{f}{E} \right)^2 = \frac{0.01300837}{0.00104649} = 12.4305$$

$$\left(\frac{f}{E} \right) = 3.5257$$

$$K_{min} = 0.4527$$

①	②	③	④	⑤			
(f/E)	0.02891247^2	2×0.16372	2×0.28187696	K			
1	0.028912	0.19263	0.8772	0.5953			
2	0.115648	0.27937	1.0572	0.4934			
3	0.260208	0.42393	1.3020	0.4564			
4	0.462592	0.6231	1.5828	0.4553			
5	0.722500	0.87652	1.8832	0.4738			
6	1.040832	1.20455	2.1975	0.5062			
7	1.416688	1.58641	2.5142	0.5411			
8	1.850368	2.01409	2.8384	0.5834			
9	2.341872	2.50559	3.1658	0.6289			
10	2.89120	3.05492	3.4956	0.6768			

$$\lambda = 1.200$$

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$$\begin{aligned} F_3 &= \left\{ 0.05626065 + 0.01049785 + 0.00057986 + 0.00034192 \right\} (f\pi^2)^2 \\ &\quad - \left\{ 0.12056327 + 0.04199140 \right\} (f\pi^2) + \left\{ 0.24112654 + 0.04199140 \right\} \\ &= 0.06762861 (f\pi^2)^2 - 0.16255467 (f\pi^2) + 0.28311794 \end{aligned}$$

$$H_1(\lambda) = 0.25 + 0.08398260 + 0.00547075 = 0.33945355$$

$$H_2(\lambda) = 0.16796560$$

$$H_3(\lambda) = 0.0625 + 0.03329109 = 0.09629109$$

$$\begin{aligned} F_1 &= - \left[0.33945355 (f\pi^2) - 0.16796560 \right] \left[0.12254000 (f\pi^2) - 0.07957143 \right] - 0.00157083 (f\pi^2)^2 \\ &= - 0.04316747 (f\pi^2)^2 + 0.04759331 (f\pi^2) - 0.01336526 \end{aligned}$$

$$\begin{aligned} F_2 &= \left[0.33945355 (f\pi^2) - 0.16796560 \right]^2 \left[0.19609603 \right] + 0.09900222 \left[0.09629109 (f\pi^2)^2 - 0.00091600 \right] \\ &= 0.00091600 \left[0.0522221 (f\pi^2)^2 - 0.11233307 (f\pi^2) + 0.0721244 \right] \\ &\quad + 0.00091600 (f\pi^2)^2 \end{aligned}$$

$$\begin{aligned} \pi^2 \frac{\sigma}{E} &= \lambda^2 \left(\frac{q}{R} \right)^2 \left\{ 0.19282196 \pi^4 f^2 - 0.41265255 \pi^2 f + 0.550682900 \right\} \frac{1}{6} \\ &\quad + \frac{(f/R)^2}{(\frac{q}{R})^2} \left\{ 0.93370370 \right\} \pi^4 \end{aligned}$$

$$K = f^2 \left\{ 0.04627727 \pi^4 f^2 - 0.09903661 f + \frac{0.13216390}{\pi^2} \right\} + \frac{0.93370370 \pi^2}{f^2}$$

$$K = \frac{\pi^2}{f^2} \left\{ 0.18510908 \left(\frac{f}{E} \right)^2 + 0.93370370 \right\} + \frac{0.13216390}{\pi^2} f^2 - 0.19807322 \left(\frac{f}{E} \right) \quad 497$$

$$= 2 \left\{ 0.02446474 \left(\frac{f}{E} \right)^2 + 0.12340192 \right\}^{\frac{1}{2}} - 0.19807322 \left(\frac{f}{E} \right)$$

$$K_0 = 0.7026 \quad 0.04892948 \left(\frac{f}{E} \right) = 0.19807322 \left\{ 0.02446474 \left(\frac{f}{E} \right)^2 + 0.12340192 \right\}$$

$$\frac{0.002394094}{0.000959825} \left(\frac{f}{E} \right)^2 = 0.00484143$$

$$\left(\frac{f}{E} \right)^2 = 3.325538$$

$$\left(\frac{f}{E} \right) = 1.823264$$

$$K_{min} = 0.5438$$

①	②	③	④	⑤	⑥	⑦
(f/E)	0.024464 ①^2	$\text{②} + 0.12340$	$2 \times \text{③}^{\frac{1}{2}}$	K		
1	0.024464	0.14786	0.7690	0.5709		
2	0.097856	0.22126	0.9408	0.5447		
3	0.220176	0.34558	1.1724	0.5182		
4	0.391424	0.51890	1.4350	0.4922		
5	0.611600	0.73500	1.7146	0.47243		

$$\lambda = 0.8020$$

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$$\begin{aligned} F_3 &= \left\{ 0.11426544 + 0.02323766 + 0.000725745 + 0.00123288 \right\} (f\pi^2)^2 \\ &\quad - \left\{ 0.61035156 + 0.09295062 \right\} (f\pi^2) + \left\{ 1.22070313 + 0.09295062 \right\} \\ &= 0.13946173 (f\pi^2)^2 - 0.70330218 (f\pi^2) + 1.31365375 \end{aligned}$$

$$H_1(\lambda) = 0.25 + 0.18590125 + 0.01972604 = 0.45562729$$

$$H_2(\lambda) = 0.37180250, \quad H_3(\lambda) = 0.0625 + 0.04644768 = 0.10894768$$

$$\begin{aligned} &- \left[0.45562729 (f\pi^2) - 0.37180250 \right] \left[0.1225400 (f\pi^2) - 0.07957143 \right] \\ &- 0.016313384 \times 0.10894768 (f\pi^2)^2 \\ &= -0.05760987 (f\pi^2)^2 + 0.08181559 (f\pi^2) - 0.02961491 = \frac{7}{4} \end{aligned}$$

$$\begin{aligned} F_2 &= 0.13302164 \left[0.45562729 (f\pi^2) - 0.37180250 \right]^2 \\ &\quad + 0.06654153 \times (0.10894768)^2 (f\pi^2)^{-1} \\ &= 0.13302164 \left\{ 0.20759623 (f\pi^2)^2 - 0.33880673 (f\pi^2) + 0.13823710 \right\} \\ &\quad + 0.00157891 (f\pi^2)^2 \end{aligned}$$

$$\begin{aligned} \pi^2 \frac{\sigma}{E} &= \left(\frac{a}{R} \right)^2 \left\{ 0.28427664 \pi^4 f^2 - 1.27978602 \pi^2 f + 1.66710703 \right\} \frac{1}{6} \\ &\quad + \frac{\left(\frac{f}{R} \right)^2}{\left(\frac{a}{R} \right)^2} \left\{ 0.95638889 \right\} \pi^4 \end{aligned}$$

$$K = \gamma^2 \left\{ 0.04737944 \pi^2 f^2 - 0.21329767 f + \frac{0.277651172}{\pi^2} \right\} + \frac{\pi^2}{\gamma^2} 0.95638889 \quad \underline{499}$$

$$= \frac{\pi^2}{\gamma^2} \left\{ 0.18951776 \left(\frac{f}{E} \right)^2 + 0.95638889 \right\} + \frac{0.277651172}{\pi^2} \gamma^2 - 0.42659534 \left(\frac{f}{E} \right)$$

$$= 2 \left\{ 0.05265773 \left(\frac{f}{E} \right)^2 + 0.26573377 \right\}^{\frac{1}{2}} - 0.42659534 \left(\frac{f}{E} \right)$$

$$K_0 = 1.0310$$

$$0.10531546 \left(\frac{f}{E} \right) = 0.42659534 \left(0.05265773 \left(\frac{f}{E} \right)^2 + 0.26573377 \right)^{\frac{1}{2}}$$

$$\frac{0.011071346}{0.009582842} \left(\frac{f}{E} \right)^2 = 32.05771 \quad \left(\frac{f}{E} \right) = 5.66195$$

$$K_{min} = 0.3802$$

①	②	③	④	⑤	⑥		
16/E							
1							
2							
3							
4							
5							
6							
7							
8							
9							
10							

$$\lambda = 0.600$$

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$$\begin{aligned} F_3 &= \left\{ 0.28940008 + 0.03379109 + 0.00082195 + 0.00262446 \right\} (f\pi^2)^2 \\ &\quad - \left\{ 1.92901235 + 0.13516436 \right\} (f\pi^2) + \left\{ 3.85802469 + 0.13516436 \right\} \\ &= 0.32663758 (f\pi^2)^2 - 2.06417671 (f\pi^2) + 3.99318905 \end{aligned}$$

$$H_1(\lambda) = 0.25 + 0.2703772 + 0.04199140 = 0.56232012, \quad H_2(\lambda) = 0.54065744$$

$$H_3(\lambda) = 0.0625 + 0.05260500 = 0.11510500$$

$$\begin{aligned} F_1 &= - \left[0.56232012 (f\pi^2) - 0.54065744 \right] \left[0.1225400 (f\pi^2) - 0.02957143 \right] \\ &\quad - 0.11510500 \times 0.016313384 (f\pi^2)^2 \\ &= -0.07078446 (f\pi^2)^2 + 0.11099678 (f\pi^2) - 0.04302089 \end{aligned}$$

$$\begin{aligned} F_2 &= 0.11619690 \left[0.56232012 (f\pi^2) - 0.54065744 \right]^2 \\ &\quad + (0.11510500)^2 (f\pi^2)^2 \times 0.05818125 \\ &= 0.11619690 \left[0.31120392 (f\pi^2)^2 - 0.60604511 (f\pi^2) + 0.29231522 \right] + 0.00072092 (f\pi^2)^2 \end{aligned}$$

$$\begin{aligned} K &= f^2 \left\{ 0.07040783 \pi^2 f^2 - 0.36428992 f + \frac{0.47809605}{\pi^2} \right\} + \frac{\pi^2}{f^2} 1.26814815 \\ &= \frac{\pi^2}{f^2} \left\{ 0.28163132 \left(\frac{f}{E}\right)^2 + 1.26814815 \right\} + \frac{0.47809605}{\pi^2} - 0.72857984 \left(\frac{f}{E}\right) \\ &= 2 \left\{ 0.13464682 \left(\frac{f}{E}\right)^2 + 0.60629662 \right\}^{\frac{1}{2}} - 0.72857984 \left(\frac{f}{E}\right) \end{aligned}$$

$$K_0 = 1.55730$$

$$0.26929364 \left(\frac{f}{E} \right) = 0.72857984 \left\{ 0.13464682 \left(\frac{f}{E} \right)^2 + 0.60629662 \right\} \frac{f}{E} \quad \underline{\underline{501}}$$

$$\frac{0.072519065}{0.071474380} \left(\frac{f}{E} \right)^2 = 308.0733 \quad \left(\frac{f}{E} \right) = 17.55202$$

$$\underline{\underline{K_{min} = 0.1869}}$$

①	②	③	④	⑤	⑥		
(B/E)							
2							
4							
6							
8							
10							
12							
14							
16							
18							
20							
22							

Consider the limiting case when $\lambda \ll 1$.

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$$\text{Then } 6\pi^2 \frac{\sigma}{E} = \frac{1}{\lambda^2} \left[\frac{17}{152} f^2 \pi^2 - \frac{3}{4} f \pi^2 + 1 \right] \frac{f^2}{\left(\frac{f}{E}\right)^2} 2 \frac{1}{\lambda^2} \pi^4$$

$$\text{or } \lambda^2 K = f^2 \left[\frac{17}{152} \pi^2 f^2 - \frac{1}{4} f + \frac{1}{6 \pi^2} \right] + \frac{\pi^2}{f^2} \frac{1}{3}$$

$$= \frac{\pi^2}{f^2} \left[\frac{17}{152} \left(\frac{f}{E}\right)^2 + \frac{1}{3} \right] + \frac{1}{6} \frac{f^2}{\pi^2} - \frac{1}{4} \left(\frac{f}{E}\right)$$

$$= 2 \left[\frac{17}{152} \left(\frac{f}{E}\right)^2 + \frac{1}{18} \right] \frac{1}{4} \left(\frac{f}{E}\right) \quad \underline{\text{O.K.}}$$

$$\frac{17}{152} \left(\frac{f}{E}\right)^2 + \frac{1}{18} = \frac{1}{64} \left(\frac{f}{E}\right)^2$$

$$\frac{1}{152} \left(\frac{f}{E}\right)^2 = \frac{1}{18} \quad \left(\frac{f}{E}\right) = \sqrt{\frac{1152}{18}} = 8$$

$$f_{\min}^2 = \pi^2 \left[34 + 2 \right] = \pi^2 36 \quad f = 6\pi$$

$$\lambda = 0.4$$

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$$\begin{aligned} F_3 &= \left\{ 1.33020020 + 0.04644768 + 0.00090289 + 0.00580941 \right\} (H\pi)^2 \\ &\quad - \left\{ 9.76562500 + 0.18579073 \right\} (H\pi) + \left\{ 19.5312500 + 0.18579073 \right\} \\ &= 1.38336018 (H\pi)^2 - 9.95141573 (H\pi) + 19.71704073 \end{aligned}$$

$$H_1(\lambda) = 0.25 + 0.37158145 + 0.09295062 = 0.71453207, \quad H_2(\lambda) = 0.74316290$$

$$H_3(\lambda) = 0.0625 + 0.05778476 = 0.12028426$$

$$\begin{aligned} F_1 &= - \left[0.71453207 (H\pi) - 0.74316290 \right] \left[0.1225400 (H\pi) - 0.07957143 \right] \\ &\quad - 0.12028426 \times 0.016313384 (H\pi)^2 \\ &= -0.08952101 (H\pi)^2 + 0.14792352 (H\pi) - 0.05913453 \end{aligned}$$

$$\begin{aligned} F_2 &= 0.09004850 \left[0.71453207 (H\pi) - 0.74316290 \right]^2 + 0.04515168 \times (0.12028426)^2 (H\pi)^2 \\ &= 0.09004850 \left[0.51055608 (H\pi)^2 - 1.0620272 (H\pi) + 0.55229110 \right] \\ &\quad + 0.00065327 (H\pi)^2 \end{aligned}$$

$$\begin{aligned} K &= \left\{ 0.142963 \frac{1}{E} \pi^2 \frac{1}{E} - 0.29193010 \frac{1}{E} + \frac{1.05102409}{\pi^2} \right\} + \frac{\pi^2}{f^2} 2.35886669 \\ &= \frac{\pi^2}{f^2} \left\{ 0.57193268 \left(\frac{f}{E} \right)^2 + 2.35886669 \right\} + \frac{1.05102409}{\pi^2} \frac{1}{E} - 1.58386020 \left(\frac{f}{E} \right) \\ &= 2 \left\{ 0.60114362 \left(\frac{f}{E} \right)^2 + 2.47936199 \right\}^{\frac{1}{2}} - 1.58386020 \left(\frac{f}{E} \right) \end{aligned}$$

$$K_0 = 3.14920,$$

$$f^2 = \pi^2 \left\{ \right.$$

$$1.20220724 \left(\frac{f}{E}\right) = 1.50220724 \left(\frac{f}{E}\right) + 2.47921199 \left(\frac{f}{E}\right)^2$$

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$$1.44549461 \quad 0.60114362 \left(\frac{f}{E}\right)^2 + 2.47936699 = 0.62715328 \left(\frac{f}{E}\right)^2$$

$$\begin{array}{r} 0.60114362 \\ 0.02600966 \\ \hline \end{array}$$

$$\left(\frac{f}{E}\right)_0 = 9.7588$$


JOURNAL OF THE AERONAUTICAL SCIENCES

31st and Northampton Streets, Easton, Pa.
Postmaster: Please send subscription orders and notices to
Bill R. C. A. Building
Rockefeller Center
New York, N. Y.

RETURN POSTAGE GUARANTEED

Printed in United States of America

Entered as second class mail matter
at the Easton, Pa. Post Office


Mr. H. T. Fan,
Aeronautics Dept.,
California Institute of Tech.,
Pasadena, California. CT

$$\frac{w}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left[1 - \left(\frac{x}{a} \right)^2 - \frac{f}{4} (1 + \cos \frac{\pi x}{a}) (1 + \cos \frac{\pi y}{b}) \right]$$

$$\frac{w_0}{R} = \frac{1}{2} \left(\frac{a}{R} \right)^2 \left[1 - \left(\frac{x}{a} \right)^2 \right]$$

$$\sigma_x = E \left\{ \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 - \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \right\}$$

$$\sigma_y = E \left\{ \frac{\partial^2 w}{\partial y^2} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right\}$$

$$\tau_{xy} = E \left\{ \frac{1}{2} \frac{\partial^2 w}{\partial x \partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right) \left(\frac{\partial w}{\partial y} \right) \right\}$$

By using the equilibrium equations: $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$, we have

$$\frac{1}{2} \frac{\partial^3 w}{\partial x^2} + \frac{\partial^3 w}{\partial y^2} = -\frac{1}{2} \left\{ \frac{\partial w}{\partial y} \left(\frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x \partial y} \right\}$$

We have

$$\frac{\partial w}{\partial x} = \left(\frac{a}{R} \right) \left\{ -\left(\frac{x}{a} \right) + \frac{f}{8} \pi \sin \frac{\pi x}{a} (1 + \cos \frac{\pi y}{b}) \right\}$$

$$\frac{\partial w}{\partial y} = \left(\frac{a}{R} \right) \frac{f}{8} \pi \left(\frac{a}{b} \right) (1 + \cos \frac{\pi x}{a}) \sin \frac{\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{R} \left[-1 + \frac{f}{8} \pi^2 \cos \frac{\pi x}{a} (1 + \cos \frac{\pi y}{b}) \right]$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{R} \left[\frac{f}{8} \pi^2 \left(\frac{a}{b} \right)^2 (1 + \cos \frac{\pi x}{a}) \cos \frac{\pi y}{b} \right]$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{R} \left[-\frac{f}{8} \pi^2 \left(\frac{a}{b} \right) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \right]$$

$$\mathcal{R} \left[\frac{\partial^2}{\partial x^2} + 2 \frac{\partial^2}{\partial y^2} \right] = - \left(\frac{q}{R} \right) \frac{1}{f} \pi \lambda \left[(1 + \cos \frac{\pi x}{a}) \sin \frac{\pi y}{b} \left\{ \frac{1}{f} \pi^2 \cos \frac{\pi x}{a} (1 + \cos \frac{\pi y}{b}) + \frac{1}{f} \pi^2 \left(\frac{q}{b} \right)^2 (1 + \cos \frac{\pi x}{a}) \cos \frac{\pi y}{b} - 1 \right\} \right.$$

$$\left. - \pi \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \right\} \frac{1}{f} \pi \sin \frac{\pi x}{a} (1 + \cos \frac{\pi y}{b}) - \frac{x}{a} \left. \right]$$

$$= - \left(\frac{q}{R} \right) \frac{1}{f} \pi \lambda \left[\frac{1}{32} \left(\lambda + \frac{1}{4} \cos \frac{\pi x}{a} + \frac{1}{4} \cos \frac{2\pi x}{a} \right) (2 \sin \frac{\pi y}{b} + \sin \frac{2\pi y}{b}) + \frac{1}{16} \left(\frac{q}{b} \right)^2 (3 + 4 \cos \frac{\pi x}{a} + \cos \frac{2\pi x}{a}) \sin \frac{\pi y}{b} \right.$$

$$\left. - (1 + \cos \frac{\pi x}{a}) \sin \frac{\pi y}{b} \right.$$

$$\left. - \frac{1}{32} \left(1 - \cos \frac{2\pi x}{a} \right) (2 \sin \frac{\pi y}{b} + \sin \frac{2\pi y}{b}) + \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \right]$$

$$= - \left(\frac{q}{R} \right) \frac{1}{f} \pi \lambda \left[\frac{1}{f} \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} + \frac{1}{f} \cos \frac{2\pi x}{a} \sin \frac{\pi y}{b} + \frac{1}{f} \cos \frac{2\pi x}{a} \sin \frac{2\pi y}{b} + \frac{1}{16} \cos \frac{2\pi x}{a} \sin \frac{\pi y}{b} \right.$$

$$\left. + \frac{1}{16} \left(\frac{q}{b} \right)^2 3 \sin \frac{\pi y}{b} + \frac{1}{4} \left(\frac{q}{b} \right)^2 \cos \frac{\pi x}{a} \sin \frac{2\pi y}{b} + \frac{1}{16} \left(\frac{q}{b} \right)^2 \cos \frac{2\pi x}{a} \sin \frac{2\pi y}{b} \right.$$

$$\left. - \sin \frac{\pi y}{b} - \cos \frac{\pi x}{a} \sin \frac{\pi y}{b} + \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \right]$$

$$\begin{aligned} \mathcal{R} \left[\frac{\partial^2}{\partial x^2} + 2 \frac{\partial^2}{\partial y^2} \right] = & - \left(\frac{a}{R} \right) \frac{f}{8} \pi h \left[- \sin \frac{\pi x}{b} + \frac{3}{16} (4\pi^2)^2 \sin^2 \frac{\pi x}{b} + \left(\frac{4\pi^2}{8} - 1 \right) \cos \frac{\pi x}{a} \sin \frac{\pi x}{b} \right. \\ & + \frac{4\pi^2}{8} \cos \frac{2\pi x}{a} \sin \frac{\pi x}{b} + \frac{4\pi^2}{16} (1 + 4h^2) \cos \frac{\pi x}{a} \sin \frac{2\pi x}{b} + \frac{4\pi^2}{16} (1 + h^2) \cos \frac{2\pi x}{a} \sin \frac{2\pi x}{b} \\ & \left. + \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a} \sin \frac{\pi x}{b} \right] \end{aligned}$$

The particular integral is

$$\begin{aligned} \mathcal{R} v = & + \left(\frac{a}{R} \right) \frac{f}{8} \pi h \left[- \frac{\sin \frac{\pi x}{b}}{2 \left(\frac{\pi}{b} \right)^2} + \frac{3}{16} (4\pi^2)^2 h^2 \frac{\sin \frac{2\pi x}{b}}{8 \left(\frac{\pi}{b} \right)^2} + \left(\frac{4\pi^2}{8} - 1 \right) \frac{\cos \frac{\pi x}{a} \sin \frac{\pi x}{b}}{\left(\frac{\pi}{a} \right)^2 + 2 \left(\frac{\pi}{b} \right)^2} \right. \\ & + \frac{4\pi^2}{8} \frac{\cos \frac{2\pi x}{a} \sin \frac{\pi x}{b}}{4 \left(\frac{\pi}{a} \right)^2 + 2 \left(\frac{\pi}{b} \right)^2} + \frac{4\pi^2}{16} (1 + 4h^2) \frac{\cos \frac{\pi x}{a} \sin \frac{2\pi x}{b}}{\left(\frac{\pi}{a} \right)^2 + 8 \left(\frac{\pi}{b} \right)^2} + \frac{4\pi^2}{16} (1 + h^2) \frac{\cos \frac{2\pi x}{a} \sin \frac{2\pi x}{b}}{4 \left(\frac{\pi}{a} \right)^2 + 8 \left(\frac{\pi}{b} \right)^2} \\ & \left. + \frac{2 \left(\frac{\pi}{a} \right)^2}{\left[\left(\frac{\pi}{a} \right)^2 + 2 \left(\frac{\pi}{b} \right)^2 \right]^2} \cos \frac{\pi x}{a} \sin \frac{\pi x}{b} + \frac{1}{\left(\frac{\pi}{a} \right)^2 + 2 \left(\frac{\pi}{b} \right)^2} \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a} \sin \frac{\pi x}{b} \right] \end{aligned}$$

$$\begin{aligned}
 \frac{v}{R} = & \left(\frac{a}{R} \right)^3 \frac{1}{8} \lambda \left[-\frac{1}{2\lambda^2} \sin \frac{\pi x}{b} + \frac{3}{128} \left(\frac{\pi}{b} \right)^2 \sin \frac{2\pi x}{b} + \left(\frac{1}{8} - 1 \right) \frac{\cos \frac{\pi x}{a} \sin \frac{\pi x}{b}}{1+2\lambda^2} \right. \\
 & + \frac{1}{16} \frac{\cos \frac{2\pi x}{a} \sin \frac{\pi x}{b}}{2+\lambda^2} + \frac{1}{16} \left(\frac{1+4\lambda^2}{1+8\lambda^2} \right) \cos \frac{\pi x}{a} \sin \frac{2\pi x}{b} + \frac{1}{64} \left(\frac{1+\lambda^2}{1+2\lambda^2} \right) \cos \frac{2\pi x}{a} \sin \frac{2\pi x}{b} \\
 & \left. + \frac{2\lambda^2}{(1+2\lambda^2)^2} \cos \frac{\pi x}{a} \sin \frac{\pi x}{b} + \frac{1}{1+2\lambda^2} \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a} \sin \frac{\pi x}{b} + a_0 \left(\frac{\pi x}{b} \right) + a_1 \cosh \frac{\sqrt{2}\lambda \pi x}{a} \sin \frac{\pi x}{b} \right]
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial v}{\partial x} = & \left(\frac{a}{R} \right)^3 \frac{1}{8} \lambda \left[\left(-\frac{1}{8} \right) \left(\frac{1+\lambda^2}{1+2\lambda^2} \right) - 1 \right] \frac{\sin \frac{\pi x}{a} \sin \frac{\pi x}{b}}{(1+2\lambda^2)^2} - \frac{1}{8} \frac{\sin \frac{2\pi x}{a} \sin \frac{\pi x}{b}}{(2+\lambda^2)} - \frac{1}{16} \left(\frac{1+4\lambda^2}{1+8\lambda^2} \right) \sin \frac{\pi x}{a} \sin \frac{2\pi x}{b} \\
 & - \frac{1}{32} \left(\frac{1+\lambda^2}{1+2\lambda^2} \right) \sin \frac{2\pi x}{a} \sin \frac{2\pi x}{b} + \frac{1}{1+2\lambda^2} \sin \frac{\pi x}{a} \sin \frac{\pi x}{b} + \frac{1}{1+2\lambda^2} \left(\frac{\pi x}{a} \right) \cos \frac{\pi x}{a} \sin \frac{\pi x}{b} \\
 & + a_1 \sqrt{2} \lambda \sinh \frac{\sqrt{2}\lambda \pi x}{a} \sin \frac{\pi x}{b} \left. \right]
 \end{aligned}$$

$$a_1 = \frac{\pi}{(1+2\lambda^2)\sqrt{2}\lambda \sinh \sqrt{2}\lambda \pi} = \frac{1}{2\lambda^2(1+2\lambda^2)} \frac{1}{\left(\frac{\sinh \sqrt{2}\lambda \pi}{\sqrt{2}\lambda \pi} \right)}$$

$$\begin{aligned}
 \frac{\partial V}{\partial f} = & \left(\frac{a}{R} \right)^2 \frac{1}{f} \left[-\frac{1}{2} \cos \frac{\pi x}{b} + \frac{3\lambda^2}{64} (f\pi^2) \cos \frac{2\pi x}{b} + \frac{\lambda^2}{(1+2\lambda^2)} \left(\frac{f\pi^2}{8} - 1 \right) \cos \frac{\pi x}{a} \cos \frac{\pi x}{b} \right. \\
 & + \frac{\lambda^2}{2+2\lambda^2} \frac{f\pi^2}{16} \cos \frac{2\pi x}{a} \cos \frac{\pi x}{b} + \frac{\lambda^2(1+4\lambda^2)}{1+8\lambda^2} \frac{f\pi^2}{8} \cos \frac{\pi x}{a} \cos \frac{2\pi x}{b} + \frac{\lambda^2(1+\lambda^2)}{1+2\lambda^2} \frac{f\pi^2}{32} \cos \frac{2\pi x}{a} \cos \frac{2\pi x}{b} \\
 & \left. + \frac{2\lambda^4}{(1+2\lambda^2)^2} \cos \frac{\pi x}{a} \cos \frac{\pi x}{b} + \frac{\lambda^2}{1+2\lambda^2} \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a} \cos \frac{\pi x}{b} + a_0 \lambda^2 + a_1 \lambda^2 \cosh \frac{\sqrt{2}\lambda \pi x}{a} \cos \frac{\pi x}{b} \right]
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{\partial V}{\partial f} \right)_{y=b} = & \left(\frac{a}{R} \right)^2 \frac{1}{f} \left[\frac{1}{2} + \frac{3\lambda^2}{64} (f\pi^2) - \frac{\lambda^2}{1+2\lambda^2} \left(\frac{f\pi^2}{8} - 1 \right) \cos \frac{\pi x}{a} - \frac{\lambda^2}{2+2\lambda^2} \frac{f\pi^2}{16} \cos \frac{2\pi x}{a} \right. \\
 & + \frac{\lambda^2(1+4\lambda^2)}{1+8\lambda^2} \frac{f\pi^2}{8} \cos \frac{\pi x}{a} + \frac{\lambda^2(1+\lambda^2)}{1+2\lambda^2} \frac{f\pi^2}{32} \cos \frac{2\pi x}{a} - \frac{2\lambda^2}{(1+2\lambda^2)^2} \cos \frac{\pi x}{a} - \frac{\lambda^2}{1+2\lambda^2} \left(\frac{\pi x}{a} \right) \sin \left(\frac{\pi x}{a} \right) \\
 & \left. + a_0 \lambda^2 - a_1 \lambda^2 \cosh \frac{\sqrt{2}\lambda \pi x}{a} \right]
 \end{aligned}$$

$$\begin{aligned}
 -\frac{\partial}{\partial E} = & \left(\frac{a}{R} \right)^2 \frac{1}{f} \left[\frac{1}{2} + \frac{3\lambda^2}{64} (f\pi^2) - \frac{\lambda^2}{1+2\lambda^2} + a_0 \lambda^2 - \frac{a_1 \lambda^2}{\sqrt{2}\lambda \pi} \sinh \sqrt{2}\lambda \pi \right] \\
 = & \left(\frac{a}{R} \right)^2 \frac{1}{f} \left[\frac{1}{2} + \frac{3}{64} \lambda^2 (f\pi^2) - \frac{\lambda^2}{1+2\lambda^2} + a_0 \lambda^2 - \frac{1}{2(1+2\lambda^2)} \right] \\
 = & \left(\frac{a}{R} \right)^2 \frac{1}{f} \left[\frac{3}{64} \lambda^2 (f\pi^2) + a_0 \lambda^2 \right]
 \end{aligned}$$

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$$\left(\frac{a}{R}\right)^2 \frac{f}{g} a_0 \lambda^2 = -\frac{\sigma}{E} - \left(\frac{a}{R}\right)^2 \frac{f}{g} \frac{3}{64} \lambda^2 (4\pi^2)$$

$$\boxed{\frac{\Delta p}{E a b t} = -\left(\frac{\sigma}{E}\right)^2 - \left(\frac{a}{R}\right)^2 \left(\frac{f}{g}\right)^2 \left(\frac{3}{64} \lambda^2 \pi^2\right) \frac{\sigma}{E}}$$

$$\begin{aligned} \frac{\partial p}{\partial y} = \left(\frac{a}{R}\right)^2 \frac{f}{g} \left[-\frac{1}{2} \cos \frac{\pi x}{b} + \frac{3}{64} \lambda^2 (4\pi^2) \cos \frac{\pi x}{b} + \frac{\lambda^2}{1+2\lambda^2} \left(\frac{\pi^2}{8} - 1 \right) \cos \frac{\pi x}{a} \cos \frac{\pi x}{b} \right. \\ \left. + \frac{\lambda^2}{2+\lambda^2} \frac{\pi^2}{16} \cos \frac{2\pi x}{a} \cos \frac{\pi x}{b} + \frac{\lambda^2 (1+4\lambda^2)}{1+8\lambda^2} \frac{\pi^2}{8} \cos \frac{\pi x}{a} \cos \frac{2\pi x}{b} + \frac{\lambda^2 (1+\lambda^2)}{1+2\lambda^2} \frac{\pi^2}{32} \cos \frac{2\pi x}{a} \cos \frac{2\pi x}{b} \right. \\ \left. + \frac{2\lambda^4}{(1+2\lambda^2)^2} \cos \frac{\pi x}{a} \cos \frac{\pi x}{b} + \frac{\lambda^2}{1+2\lambda^2} \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a} \cos \frac{\pi x}{b} - \frac{3}{64} \lambda^2 (4\pi^2) + a \lambda^2 \cosh \frac{\pi x}{a} \cos \frac{\pi x}{b} \right] - \frac{\sigma}{E} \end{aligned}$$

$$\frac{1}{2} \left(\frac{\partial a}{\partial y} \right)^2 = \left(\frac{a}{R} \right)^2 \frac{f}{g} \left[\frac{\pi^2}{64} \lambda^2 \left\{ 3 + 4 \cos \frac{\pi x}{a} + \cos \frac{2\pi x}{a} - 3 \cos \frac{\pi x}{b} + 4 \cos \frac{\pi x}{a} \cos \frac{\pi x}{b} - \cos \frac{2\pi x}{a} \cos \frac{2\pi x}{b} \right\} \right]$$

$$\begin{aligned} \frac{\partial p}{\partial x} = \left(\frac{a}{R}\right)^2 \frac{f}{g} \left[\left\{ \frac{\pi^2}{16} \lambda^2 \cos \frac{\pi x}{a} + \frac{\pi^2}{64} \lambda^2 \cos \frac{2\pi x}{a} \right\} \right. \\ \left. + \left\{ -\frac{1}{2} + \left(\frac{\pi^2}{8} - \frac{\lambda^2}{(1+2\lambda^2)^2} \right) \cos \frac{\pi x}{a} + \frac{\lambda^2}{2+\lambda^2} \frac{\pi^2}{16} \cos \frac{2\pi x}{a} + \frac{\lambda^2}{1+2\lambda^2} \left(\frac{\pi x}{a} \right) \sin \frac{\pi x}{a} \right\} \right. \\ \left. + \left\{ \frac{\lambda^2}{1+2\lambda^2} \frac{\pi^2}{16} \cos \frac{\pi x}{a} + \frac{\lambda^2}{1+2\lambda^2} \frac{\pi^2}{64} \cos \frac{2\pi x}{a} \right\} \cos \frac{2\pi x}{b} \right] - \frac{\sigma}{E} \end{aligned}$$

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$$\begin{aligned}
\frac{1}{2E^2ab} \int_0^{\infty} \int_0^b \phi_y^2 dx dy &= \left(\frac{a}{b}\right)^4 \left(\frac{1}{8}\right)^2 \left[\frac{1}{4} \left(\frac{16\lambda^2}{16}\right)^2 + \left(\frac{16\lambda^2}{64}\lambda^2\right)^2 \frac{1}{4} + \frac{1}{8} \left(\frac{\lambda^2}{1+2\lambda^2} + \frac{16\lambda^2}{16}\right)^2 + \frac{1}{8} \left(\frac{\lambda^2}{1+2\lambda^2} + \frac{16\lambda^2}{64}\right)^2 \right. \\
&+ \frac{1}{16} + \frac{1}{8} \left(\frac{16\lambda^2}{8} \frac{\lambda^2}{1+2\lambda^2} - \frac{\lambda^2}{(1+2\lambda^2)^2} \right)^2 + \frac{1}{8} \left(\frac{\lambda^2}{2+2\lambda^2} - \frac{16\lambda^2}{16} \right)^2 \\
&- \frac{1}{4} \int_0^1 \left\{ \frac{\lambda^2}{1+2\lambda^2} \frac{16\lambda^2}{(a)} \sin \frac{16\lambda^2}{a} + a_1 \lambda^2 \cos \frac{\sqrt{2}\lambda^2 16\lambda^2}{a} \right\} d\left(\frac{x}{a}\right) \\
&+ \frac{1}{2} \left\{ \frac{16\lambda^2}{8} \frac{\lambda^2}{1+2\lambda^2} - \frac{\lambda^2}{(1+2\lambda^2)^2} \right\} \int_0^1 \cos \frac{16\lambda^2}{a} \left\{ -\frac{\lambda^2}{1+2\lambda^2} \frac{16\lambda^2}{(a)} \sin \frac{16\lambda^2}{a} + a_1 \lambda^2 \cos \frac{\sqrt{2}\lambda^2 16\lambda^2}{a} \right\} d\left(\frac{x}{a}\right) \\
&+ \frac{1}{2} \frac{\lambda^2}{2+2\lambda^2} \frac{16\lambda^2}{16} \int_0^1 \cos \frac{2\lambda^2}{a} \left\{ \frac{\lambda^2}{1+2\lambda^2} \frac{16\lambda^2}{(a)} \sin \frac{16\lambda^2}{a} + a_1 \lambda^2 \cos \frac{\sqrt{2}\lambda^2 16\lambda^2}{a} \right\} d\left(\frac{x}{a}\right) \\
&+ \frac{1}{4} \int_0^1 \left\{ -\frac{\lambda^2}{1+2\lambda^2} \frac{16\lambda^2}{(a)} \sin \frac{16\lambda^2}{a} + a_1 \lambda^2 \cos \frac{\sqrt{2}\lambda^2 16\lambda^2}{a} \right\}^2 d\left(\frac{x}{a}\right) \Big]
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2E'ab} \int_0^a \int_0^b \phi_y^2 dx dy = \left(\frac{b^4}{8} \right) \left(\frac{a^2}{8} \right)^2 \left[\frac{17}{1384} (A\pi)^2 b^4 + \frac{1}{2048} \left(\frac{b^2}{1+4b^2} \right)^2 (A\pi)^2 + \frac{1}{32768} \left(\frac{b^2}{1+4b^2} \right)^2 (4b^2)^2 + \frac{1}{16} \right. \\
& + \frac{1}{512} \left(\frac{b^2}{1+4b^2} \right)^2 (A\pi)^2 + \frac{1}{2048} \left(\frac{b^2}{2+4b^2} \right)^2 (4b^2)^2 - \frac{1}{32} \frac{b^4}{(1+4b^2)^3} (A\pi)^2 + \frac{1}{8} \frac{b^4}{(1+4b^2)^4} \\
& - \frac{1}{4} \frac{b^2}{1+4b^2} - \frac{1}{4} \frac{1}{2(1+4b^2)} + \left\{ \frac{A^2}{8} \frac{b^2}{1+4b^2} - \frac{b^2}{(1+4b^2)^2} \right\} \left\{ -\frac{1}{8} \frac{b^2}{1+4b^2} - \frac{1}{2} \frac{b^2}{(1+4b^2)^2} \right\} \\
& + \frac{b^2}{2+4b^2} \frac{A^2}{32} \left\{ -\frac{1}{3} \frac{b^2}{1+4b^2} + \frac{b^2}{(1+4b^2)^2(2+b^2)} + \frac{1}{4} \left(\frac{\pi^2}{6} - \frac{1}{4} \right) \left(\frac{b^2}{1+4b^2} \right)^2 \right\} \\
& + \frac{b^2}{8(1+4b^2)^2} \left(\frac{\sinh \sqrt{2} \pi}{\sqrt{2} \pi} \right) \left\{ \frac{1}{\cosh \sqrt{2} \pi} - \frac{4b^2}{(1+4b^2)^2} \left(\frac{\sinh \sqrt{2} \pi}{\sqrt{2} \pi} \right) \right\} \\
& + \frac{1}{4} \frac{1}{4(1+4b^2)^2} \frac{1}{\left(\frac{\sinh \sqrt{2} \pi}{\sqrt{2} \pi} \right)^2} \left\{ \frac{1}{2} + \frac{1}{4\sqrt{2} \pi} \sinh 2\sqrt{2} \pi \right\} \left[-\frac{1}{2} \left(\frac{b^2}{E} \right)^2 \right. \\
& \left. \frac{1}{2E'ab} \int_0^a \int_0^b \phi_x^2 dx dy = \left(\frac{b^4}{8} \right) \left(\frac{a^2}{8} \right)^2 \left[\frac{105}{32768} (A\pi)^2 - \frac{5}{48} (A\pi)^2 + \left(\frac{\pi^2}{8} - \frac{3}{16} \right) \right] \right]
\end{aligned}$$

$$\begin{aligned}
 \frac{\partial \psi}{\partial x} = & \left(\frac{a^2}{R} \right) \frac{1}{g} \lambda^2 \left[2 \left(\frac{1+\lambda^2}{(1+\lambda^2)^2} \right) - \frac{1}{1+\lambda^2} \right] \left\{ \sin \frac{\pi x}{a} \sin \frac{\pi y}{g} - \frac{1}{g(2+\lambda^2)} \left(\frac{\pi^2}{g} \right) \sin \frac{2\pi x}{a} \sin \frac{\pi y}{g} \right. \\
 & - \left. \frac{(1+\lambda^2)}{(1+\lambda^2)^2} \left(\frac{\pi^2}{16} \right) \sin \frac{\pi x}{a} \sin \frac{2\pi y}{g} - \left(\frac{1+\lambda^2}{(1+\lambda^2)^2} \right) \frac{\pi^2}{32} \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{g} + \frac{1}{1+\lambda^2} \left(\frac{\pi^2}{a} \right) \cos \frac{\pi x}{a} \sin \frac{\pi y}{g} \right. \\
 & \left. + a \sqrt{2} \lambda \sin \frac{\sqrt{2} \lambda \pi x}{a} \sin \frac{\pi y}{g} \right]
 \end{aligned}$$

$$\frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} = \left(\frac{a^2}{R} \right) \frac{1}{g} \lambda^2 \left\{ - \left(\frac{\pi^2}{a} \right) \sin \frac{\pi y}{g} - \left(\frac{\pi^2}{a} \right) \cos \frac{\pi x}{a} \sin \frac{\pi y}{g} + \frac{\pi^2}{32} \left(2 \sin \frac{\pi x}{a} + \sin \frac{2\pi x}{a} \right) \left(2 \sin \frac{\pi y}{g} + \sin \frac{2\pi y}{g} \right) \right\}$$

$$\begin{aligned}
 = & \left(\frac{a^2}{R} \right) \frac{1}{g} \lambda^2 \left\{ - \left(\frac{\pi^2}{a} \right) \sin \frac{\pi y}{g} - \left(\frac{\pi^2}{a} \right) \cos \frac{\pi x}{a} \sin \frac{\pi y}{g} + \frac{\pi^2}{g} \sin \frac{\pi x}{a} \sin \frac{\pi y}{g} + \frac{\pi^2}{16} \sin \frac{2\pi x}{a} \sin \frac{\pi y}{g} \right. \\
 & \left. + \frac{\pi^2}{16} \sin \frac{\pi x}{a} \sin \frac{2\pi y}{g} + \frac{\pi^2}{32} \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{g} \right\}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial^2 \psi}{\partial x \partial y} = & \left(\frac{a^2}{R} \right) \frac{1}{g} \lambda^2 \left\{ - \left(\frac{\pi^2}{a} \right) \sin \frac{\pi y}{g} - \frac{2\lambda^2}{1+\lambda^2} \left(\frac{\pi^2}{a} \right) \cos \frac{\pi x}{a} \sin \frac{\pi y}{g} + 2 \left(\frac{1+\lambda^2}{(1+\lambda^2)^2} \right) \left(\frac{\pi^2}{g} \right) \sin \frac{\pi x}{a} \sin \frac{\pi y}{g} \right. \\
 & + \frac{\lambda^2}{16(2+\lambda^2)} \left(\frac{\pi^2}{a} \right) \sin \frac{2\pi x}{a} \sin \frac{\pi y}{g} + \frac{\lambda^2}{4(1+\lambda^2)} \left(\frac{\pi^2}{a} \right) \sin \frac{\pi x}{a} \sin \frac{2\pi y}{g} + \frac{\lambda^2}{32} \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{g} \\
 & \left. + \frac{a_1 \sqrt{2} \lambda \sin \frac{\sqrt{2} \lambda \pi x}{a}}{2} \sin \frac{\pi y}{g} \right\}
 \end{aligned}$$

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$$\frac{E_{xy}}{E} = \frac{a^2}{R} \frac{1}{8} \left[\left\{ -\frac{\lambda}{2} \frac{D(\lambda)}{a} - \frac{\lambda^3}{1+2\lambda^2} \frac{D(\lambda)}{a} \cos \frac{\pi x}{a} + \left\{ \frac{\lambda(1+\lambda^2)}{(1+2\lambda^2)^2} + \frac{\lambda^3}{1+2\lambda^2} \frac{D(\lambda)}{8} \right\} \sin \frac{\pi x}{a} + \frac{\lambda^3}{32(2+\lambda^2)} \frac{D(\lambda)}{8} \sin \frac{2\pi x}{a} \right\} \right. \\ \left. + \frac{a}{2} \sqrt{2} \lambda \sinh \frac{\sqrt{2} \lambda \pi x}{a} \sin \frac{\pi x}{8} \right]$$

$$+ \left\{ \frac{\lambda^3}{8(1+\lambda^2)} \frac{D(\lambda)}{a} \sin \frac{\pi x}{a} + \frac{\lambda^3}{(1+\lambda^2)} \frac{D(\lambda)}{64} \sin \frac{2\pi x}{a} \right\} \sin \frac{2\pi x}{8}$$

$$\frac{1}{E_{z0b}} \int_0^{a/2} \int_0^{a/2} dz dy = \left(\frac{a}{R} \right)^4 \left(\frac{D}{8} \right)^2 \left[\frac{1}{4} \left\{ \frac{\lambda(1+\lambda^2)}{(1+2\lambda^2)^2} + \frac{\lambda^3}{1+2\lambda^2} \frac{D(\lambda)}{8} \right\} + \frac{1}{4} \left\{ \frac{\lambda^3}{32(2+\lambda^2)} \frac{D(\lambda)}{8} \right\}^2 \right]$$

$$+ \frac{1}{4} \left\{ \frac{\lambda^3}{8(1+\lambda^2)} \frac{D(\lambda)}{a} \right\}^2 + \frac{1}{4} \left\{ \frac{\lambda^3}{1+2\lambda^2} \frac{D(\lambda)}{64} \right\}^2$$

$$+ \left\{ \frac{\lambda(1+\lambda^2)}{(1+2\lambda^2)^2} + \frac{\lambda^3}{1+2\lambda^2} \frac{D(\lambda)}{8} \right\} \int_0^1 \sin \frac{\pi x}{a} \left\{ -\frac{\lambda}{2} \frac{D(\lambda)}{a} - \frac{\lambda^3}{1+2\lambda^2} \frac{D(\lambda)}{a} \cos \frac{\pi x}{a} + \frac{a}{2} \sqrt{2} \lambda \sinh \frac{\sqrt{2} \lambda \pi x}{a} \right\} d\left(\frac{x}{a}\right)$$

$$+ \frac{\lambda^3}{32(2+\lambda^2)} \left(\frac{D(\lambda)}{8} \right)^2 \int_0^1 \sin \frac{2\pi x}{a} \left\{ -\frac{\lambda}{2} \frac{D(\lambda)}{a} - \frac{\lambda^3}{1+2\lambda^2} \frac{D(\lambda)}{a} \cos \frac{\pi x}{a} + \frac{a}{2} \sqrt{2} \lambda \sinh \frac{\sqrt{2} \lambda \pi x}{a} \right\} d\left(\frac{x}{a}\right)$$

$$+ \frac{1}{2} \int_0^1 \left\{ -\frac{\lambda}{2} \frac{D(\lambda)}{a} - \frac{\lambda^3}{1+2\lambda^2} \frac{D(\lambda)}{a} \cos \frac{\pi x}{a} + \frac{a}{2} \sqrt{2} \lambda \sinh \frac{\sqrt{2} \lambda \pi x}{a} \right\}^2 d\left(\frac{x}{a}\right)$$

APP

$$\begin{aligned}
& \frac{1}{E_{ab}} \int_0^a \int_0^b xy^2 dx dy = \frac{(b^4)}{8} \left(\frac{a^2}{2} \right) \left[\frac{\lambda^2 (1+\lambda^2)^2}{4(1+\lambda^2)^4} + \frac{\lambda^4 (1+\lambda^2)}{16(1+\lambda^2)^3} f(\pi^2) + \frac{\lambda^6}{256(1+\lambda^2)^2} f(\pi^2)^2 \right] \\
& + \frac{\lambda^6}{4096(2+\lambda^2)^2} f(\pi^2)^2 + \frac{\lambda^6}{256(1+\lambda^2)^2} f(\pi^2)^2 + \frac{\lambda^6}{16384(1+\lambda^2)^2} f(\pi^2)^2 \\
& + \left\{ \frac{\lambda(1+\lambda^2)}{(1+\lambda^2)^2} + \frac{\lambda^3}{1+\lambda^2} \frac{f(\pi^2)}{8} \right\} \left\{ -\frac{2}{2} + \frac{1}{4} \frac{\lambda^3}{1+\lambda^2} + \frac{1}{2} \frac{\lambda}{(1+\lambda^2)^2} \right\} \\
& + \frac{\lambda^3}{32(2+\lambda^2)} f(\pi^2) \left\{ +\frac{2}{4} - \frac{2}{3} \frac{\lambda^3}{1+\lambda^2} - \frac{1}{2} \frac{\lambda}{(1+\lambda^2)(2+\lambda^2)} \right\} \\
& + \frac{1}{2} \left\{ \frac{\lambda^2}{4} \frac{\pi^2}{3} - \frac{2\lambda^4}{1+\lambda^2} - \frac{1}{4} \frac{1}{(1+\lambda^2)} \frac{1}{\sinh \sqrt{2}\lambda\pi} \left(\sqrt{2}\lambda\pi \cosh \sqrt{2}\lambda\pi - \sinh \sqrt{2}\lambda\pi \right) \right. \\
& + \frac{\lambda^6}{(1+\lambda^2)^2} \left(\frac{\pi^2}{6} + \frac{1}{4} \right) - \frac{\lambda^4 \pi}{(1+\lambda^2)^2 \sinh \sqrt{2}\lambda\pi} \left(-\frac{\sqrt{2}\lambda \cosh \sqrt{2}\lambda\pi}{1+\lambda^2} + \frac{1}{\pi} \frac{(2\lambda^2-1) \sinh \sqrt{2}\lambda\pi}{(1+\lambda^2)^2} \right) \\
& \left. + \frac{\pi^2 \lambda^2}{4(1+\lambda^2)^2 \sinh \sqrt{2}\lambda\pi} \left(\frac{\sinh \sqrt{2}\lambda\pi}{4\sqrt{2}\lambda\pi} - \frac{1}{2} \right) \right\}
\end{aligned}$$

$$\frac{E_1}{E_{00}t} = \frac{1}{10} \left[\mathcal{H}_1(u) (H^2) - \mathcal{H}_2(u) (H^2) + \mathcal{H}_3(u) \right] + \frac{1}{2} \left(\frac{v}{E} \right)^2$$

$$\mathcal{H}_1(u) = \frac{105}{32768} + \frac{172^4}{16384} + \frac{17^4}{2048(1+u)^2} + \frac{1^4}{32768(1+2u)^2} + \frac{1^4}{512(1+2u)^2} + \frac{1^4}{2048(2+u)^2}$$

$$+ \frac{1^6}{256(1+2u)^2} + \frac{1^6}{4096(2+u)^2} + \frac{1^6}{256(1+u)^2} + \frac{1^6}{16384(1+2u)^2}$$

$$= \frac{105}{32768} + \frac{172^4}{16384} + \frac{1^4}{2048(1+u)^2} + \frac{1^4}{32768(1+2u)^2} + \frac{1^4}{512(1+2u)^2} + \frac{1^4}{4096(2+u)^2}$$

$$\mathcal{H}_1 = \frac{105}{32768} + \frac{172^4}{16384} + \frac{1^4}{2048(1+u)^2} + \frac{1^4}{32768(1+2u)^2} + \frac{1^4}{4096(2+u)^2} + \frac{1^4}{512(1+2u)^2}$$

$$\begin{aligned}
 u_2(\lambda) &= \frac{5}{48} + \frac{1}{32} \frac{\lambda^4}{(1+2\lambda^2)^3} + \frac{1}{64} \frac{\lambda^4}{(1+2\lambda^2)^2} + \frac{1}{16} \frac{\lambda^4}{(1+2\lambda^2)} + \frac{1}{96} \frac{\lambda^4}{(1+2\lambda^2)(1+2\lambda^2)} \\
 &\quad - \frac{1}{32} \frac{\lambda^4}{(1+2\lambda^2)(2+1\lambda^2)^2} + \frac{1}{16} \frac{\lambda^4(1+2\lambda^2)^4}{(1+2\lambda^2)^3} + \frac{1}{16} \frac{\lambda^4}{(1+2\lambda^2)^2} - \frac{1}{32} \frac{\lambda^4}{(1+2\lambda^2)^2} - \frac{1}{16} \frac{\lambda^4}{(1+2\lambda^2)^3} \\
 &\quad - \frac{1}{128} \frac{\lambda^4}{(2+2\lambda^2)} + \frac{1}{48} \frac{\lambda^4}{(2+2\lambda^2)(1+2\lambda^2)} + \frac{1}{64} \frac{\lambda^4}{(1+2\lambda^2)(2+1\lambda^2)^2}
 \end{aligned}$$

$$u_2 = \frac{5}{48} + \frac{3}{64} \frac{\lambda^4}{(1+2\lambda^2)} + \frac{\lambda^4}{384(2+1\lambda^2)} - \frac{1}{64} \frac{\lambda^4}{(1+2\lambda^2)(2+1\lambda^2)^2}$$

$$\begin{aligned}
 u_3(\lambda) &= \frac{\pi^2}{8} - \frac{3}{16} + \frac{1}{16} + \frac{1}{8} \frac{\lambda^4}{(1+2\lambda^2)^4} - \frac{1}{4} \frac{\lambda^4}{(1+2\lambda^2)^2} - \frac{1}{8} \frac{1}{(1+2\lambda^2)} + \frac{1}{8} \frac{\lambda^4}{(1+2\lambda^2)^3} + \frac{1}{8} \frac{\lambda^4}{(1+2\lambda^2)^4} \\
 &\quad + \frac{1}{4} \left(\frac{\pi^2}{6} - \frac{1}{4} \right) \frac{\lambda^4}{(1+2\lambda^2)^2} + \frac{1}{4} \frac{\lambda^2}{(1+2\lambda^2)^3} - \frac{\cos \sqrt{2}\pi}{\left(\frac{\sin \sqrt{2}\pi}{\sqrt{2}\pi} \right)^4} - \frac{\lambda^4}{(1+2\lambda^2)^4} \\
 &\quad + \frac{1}{16(1+2\lambda^2)^2} \left[\frac{\frac{1}{2} + \frac{1}{4\sqrt{2}\pi} \sin 2\sqrt{2}\pi}{\left(\frac{\sin \sqrt{2}\pi}{\sqrt{2}\pi} \right)^2} + \frac{\lambda^2(1+2\lambda^2)^2}{4(1+2\lambda^2)^4} - \frac{1}{2} \frac{\lambda^2(1+2\lambda^2)}{(1+2\lambda^2)^2} + \frac{1}{4} \frac{\lambda^4(1+2\lambda^2)}{(1+2\lambda^2)^3} \right. \\
 &\quad \left. + \frac{1}{2} \frac{\lambda^2(1+2\lambda^2)}{(1+2\lambda^2)^4} + \frac{\lambda^2}{8} - \frac{\lambda^4}{8(1+2\lambda^2)^2} - \frac{1}{8} \frac{1}{(1+2\lambda^2)} - \frac{\cos \sqrt{2}\pi}{\left(\frac{\sin \sqrt{2}\pi}{\sqrt{2}\pi} \right)^4} + \frac{1}{8} \frac{1}{(1+2\lambda^2)^2} + \frac{1}{2(1+2\lambda^2)^2} \left(\frac{\pi^2}{6} + \frac{1}{4} \right) \right. \\
 &\quad \left. + \frac{1}{2} \frac{\lambda^4}{(1+2\lambda^2)^3} - \frac{\cos \sqrt{2}\pi}{\left(\frac{\sin \sqrt{2}\pi}{\sqrt{2}\pi} \right)^4} + \frac{1}{2} \frac{\lambda^4(1-2\lambda^2)}{(1+2\lambda^2)^4} + \frac{1}{2} \frac{1}{16(1+2\lambda^2)^2} + \frac{1}{4\sqrt{2}\pi} \frac{\sin 2\sqrt{2}\pi}{\left(\frac{\sin \sqrt{2}\pi}{\sqrt{2}\pi} \right)^2} \right]
 \end{aligned}$$

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$$H_3(\lambda) = -\frac{1}{16(1+2\lambda)^2} + \frac{\sqrt{\frac{\cos \sqrt{2}\lambda\pi}{\left(\frac{\sin \sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)}}}{\frac{\sqrt{2}}{8} - \frac{1}{8} + \frac{1}{4}\left(\frac{\pi^2 - \frac{1}{4}}{6}\right)} + \frac{1}{2}\left(\frac{\pi^2 - \frac{1}{4}}{6}\right)^2 + \frac{1}{2}\left(\frac{\pi^2 - \frac{1}{4}}{6}\right)\frac{\lambda^6}{(1+2\lambda)^2}$$

$$+ \frac{3}{8} \frac{\lambda^2(2-\lambda^2)}{(1+2\lambda)^3} - \frac{1}{8} + \frac{1}{8} \frac{\lambda^4(3+2\lambda^2)}{(1+2\lambda)^3} - \frac{1}{2} \frac{\lambda^2(1+\lambda^2)}{(1+2\lambda)^2} + \frac{\lambda^2\pi^2 + 1}{24} \frac{(1-6\lambda^4)}{(1+2\lambda)^2}$$

$$= -\frac{1}{16(1+2\lambda)^2} + \frac{\sqrt{\frac{\cos \sqrt{2}\lambda\pi}{\left(\frac{\sin \sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)}}}{\frac{\sqrt{2}}{8} - \frac{1}{4} + \frac{\pi^2}{24}} + \frac{\lambda^4}{24} \frac{\lambda^2}{(1+2\lambda)^2} + \frac{\pi^2}{24} \lambda^2 - \frac{1}{16} \frac{\lambda^4(1-2\lambda^2)}{(1+2\lambda)^2}$$

$$+ \frac{1}{4} \frac{\lambda^2(3+\lambda^4)}{(1+2\lambda^2)^3} - \frac{1}{2} \frac{\lambda^2(1+\lambda^2)}{(1+2\lambda)^2} + \frac{1}{8} - \frac{1}{4} \frac{\lambda^2(1+4\lambda^2)}{(1+2\lambda)^2}$$

$$= -\frac{1}{16(1+2\lambda)^2} + \frac{\cos \sqrt{2}\lambda\pi}{\left(\frac{\sin \sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} + \frac{\pi^2}{8} \left\{ 1 + \frac{1}{3} \frac{\lambda^2(1+\lambda^2)}{1+2\lambda^2} \right\} - \frac{1}{8} - \frac{1}{16} \frac{\lambda^4}{(1+2\lambda)^2}$$

$$+ \frac{1}{4} \frac{\lambda^6}{(1+2\lambda)^2} - \frac{1}{2} \frac{\lambda^2(1+\lambda^2)}{(1+2\lambda)^2} + \frac{1}{4} \frac{\lambda^2(3+\lambda^4)}{(1+2\lambda)^2} - \frac{1}{4} \frac{\lambda^2(1+4\lambda^2)}{(1+2\lambda)^2}$$

$$H_3 = \frac{1}{8}(\pi^2 - 1) - \frac{1}{16(1+2\lambda)^2} - \frac{\cos \sqrt{2}\lambda\pi}{\left(\frac{\sin \sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} + \frac{\pi^2}{24} \frac{\lambda^2(1+\lambda^2)}{(1+2\lambda)^2} - \frac{1}{16} \frac{\lambda^2(4+17\lambda^2)}{(1+2\lambda)^2} + \frac{\lambda^2(\lambda^4 - 2\lambda^2 - 2)}{4(1+2\lambda)^2}$$

$$+ \frac{1}{4} \frac{\lambda^2(3+\lambda^4)}{(1+2\lambda)^3}$$

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$$\begin{aligned}
\frac{P}{abtE} &= \left(\frac{a^4}{R}\right) \left(\frac{f}{f}\right)^2 \left[H_1 H_2^2 - \frac{f}{2} H_1 H_2 + H_3 \right] + \left(\frac{a^9}{R}\right) \left(\frac{f}{f}\right)^2 \pi^4 \left[\frac{1}{32} (1+\lambda^4) + \frac{1}{48} \lambda^2 \right] \\
&- \left(\frac{a^2}{R}\right) \left(\frac{f}{f}\right)^2 \left(\frac{3}{8} \lambda^2 \pi^2\right) \frac{\sigma}{E} - \frac{f(E)}{2(E)} \\
\frac{3}{4} \lambda^2 \pi^2 \frac{\sigma}{E} &= \left(\frac{a^9}{R}\right)^2 \left[4 H_1 (1/\pi^2) - 3 H_2 (H_2^2) + 2 H_3 \right] + \left(\frac{f}{R}\right)^2 \pi^4 \left[\frac{1}{16} (1+\lambda^4) + \frac{1}{24} \lambda^2 \right] \\
\lambda^2 K &= f^2 \left[\frac{16}{3} H_1 f^2 \pi^2 - 4 H_2 f + \frac{f}{3} H_3 \frac{1}{\pi^2} \right] + \frac{\pi^2}{f^2} \left[\frac{1}{12} (1+\lambda^4) + \frac{1}{18} \lambda^2 \right] \\
&= \frac{\pi^2}{f^2} \left[\frac{64}{3} H_1 \left(\frac{f}{E}\right)^2 + \left\{ \frac{1}{12} (1+\lambda^4) + \frac{1}{18} \lambda^2 \right\} + \frac{f}{3} \frac{\pi^2}{\pi^2} H_3 - 8 H_2 \left(\frac{f}{E}\right) \right] \\
&= 2 \left[\frac{512}{9} H_1 H_3 \left(\frac{f}{E}\right)^2 + H_3 \left\{ \frac{1}{9} (1+\lambda^4) + \frac{1}{27} \lambda^2 \right\} - 8 H_2 \left(\frac{f}{E}\right) \right]
\end{aligned}$$

$$\begin{aligned} \frac{512}{9} H_1 &= \frac{105}{64 \times 8_3} + \frac{17}{32 \times 9} + \frac{1}{4 \times 81} + \frac{65}{64 \times 27} + \frac{1}{8 \times 27} \\ &= 0.1822917 + 0.0590278 + 0.0030864 + 0.0376157 + 0.0046296 \\ &= 0.2866512 \end{aligned}$$

$$\begin{aligned} 8H_2 &= \frac{40}{48} + \frac{3}{8} \frac{1}{3} + \frac{1}{48} \frac{1}{3} - \frac{1}{8} \frac{1}{27} \\ &= 0.8333333 + 0.1250000 + 0.00694444 - 0.0046296 \\ &= 0.9606481 \end{aligned}$$

$$\begin{aligned} H_3 &= 1.1087006 - \frac{4.444113}{16 \times 9} + \frac{\pi^2}{24} \frac{4}{3} - \frac{1}{16} \frac{21}{3} - \frac{1}{4} \frac{3}{9} + \frac{1}{27} \\ &= 1.1087006 - 0.0308619 + 0.5498311 - 0.4375 - 0.0833333 + 0.0370370 \\ &= 1.1438735 \end{aligned}$$

$$\begin{aligned} K &= 2 \left[0.3278927 \left(\frac{f}{E} \right)^2 + 0.6776509 \right]^{\frac{1}{2}} - 0.9606481 \left(\frac{f}{E} \right) \\ 0.6557654 \left(\frac{f}{E} \right) &= 0.9606481 \left[0.3278927 \left(\frac{f}{E} \right)^2 + 0.6776509 \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \frac{0.4300545 \left(\frac{f}{E} \right)^2}{0.3025940} &= 0.6255512 \\ \frac{0.1274505}{0.3025940} \left(\frac{f}{E} \right)^2 &= 4.908109 \end{aligned}$$

$$\left(\frac{f}{E} \right) = 2.215444$$

$$K_{\min} = 0.8965$$

$$K_0 = 1.646636$$

$$\lambda = 0.5$$

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$$\frac{57.9}{9} H_1 = 0.1822917 + \frac{17 \times 0.0625}{288} + \frac{0.0625}{108} + \frac{65}{864} \frac{0.0625}{864} + \frac{1}{162} \frac{0.0625}{162}$$

$$= 0.1822917 + 0.0625 (0.05952777 + 0.00925926 + 0.07523148 + 0.00617284) \\ = 0.1822917 + 0.0093557 = 0.1916474 \quad 0.7665896$$

$$fH_2 = 0.6333333 + 0.0625 \left(\frac{3}{8} \frac{1}{1.5} + \frac{1}{48} \frac{1}{2.25} - \frac{1}{8} \frac{1}{1.5 \times 2.25^2} \right) \\ = 0.6333333 + 0.0625 (0.25 + 0.00925926 - 0.01646091) \\ = 0.6485062 \quad 3.3940328$$

$$H_3 = 1.1087006 - \frac{1}{36} 2.274322 + \frac{\pi^2}{36} \frac{0.25 \times 1.75}{36} - \frac{1}{24} \frac{0.25 \times 1.25}{24} \\ - \frac{0.25 \times 2.4325}{9} + \frac{0.25 \times 3.0625}{13.5}$$

$$= 1.1087006 - 0.06312561 + 0.199431 - 0.08593750 - 0.06770833 + 0.05671296 \\ = 1.1471353 \quad 4.5905412$$

$$K = 2 \left[3.5190611 \left(\frac{f}{E} \right)^2 + 5.0155909 \right]^{\frac{1}{2}} - 3.3940328 \left(\frac{f}{E} \right)$$

$$\frac{49.5351641}{40.5346787} \left(\frac{f}{E} \right)^2 = \frac{57.7766917}{6.421448} \quad \left(\frac{f}{E} \right) = 2.534058$$

$$8.9974854$$

$$K_{min} = 1.4 - -$$

[G e n e r a l I n f o r m a t i o n]

书名= 钱学森力学手稿 4 英文

作者= 钱学森著

页数= 1 4 9

S S 号= 1 3 2 4 6 0 1 1

出版日期= 2 0 1 2 . 0 3